Pergamon

PII: S1352-2310(96)00148-3

Atmospheric Environment Vol. 31, No. 1, pp. 59–71, 1997

1352-2310/97 $15.00 + 0.00

A GENERALIZED MATHEMATICAL SCHEME TO ANALYTICALLY SOLVE THE ATMOSPHERIC DIFFUSION EQUATION WITH DRY DEPOSITION

JIN-SHENG LIN and LYNN M. HILDEMANN*

Department of Civil Engineering, Stanford University, Stanford, CA 94305-4020, U.S.A.

(*First received* 14 *September* 1995 *and in final form* 2 *May* 1996)

Abstract—A generalized mathematical scheme is developed to simulate the turbulent dispersion of pollutants which are adsorbed or deposit to the ground. The scheme is an analytical (exact) solution of the atmospheric diffusion equation with height-dependent wind speed and eddy diffusivities, and with a Robin-type boundary condition at the ground. Unlike published solutions of similar problems where complex or non-programmable (e.g., hypergeometric or Kummer) functions are obtained, the analytical solution proposed herein consists of two previously derived Green's functions (modified Bessel functions) expressed in an integral form that is amenable to numerical integration. In the case of invariant wind speed and turbulent eddies with height (i.e., Gaussian deposition plume), the solution reduces to an equivalent well-known heat conduction solution. The physical behavior represented by the Green's functions comprising the solution can be interpreted. This generalized scheme can be modified further to account for inversion effects or other meteorological conditions. The solution derived is useful for examining the accuracy and performance of sophisticated numerical dispersion models, and is particularly suitable for modeling the transport of pollutants undergoing strong surface adsorption or high depositional losses. Copyright © 1996 Elsevier Science Ltd

Key word index: Analytical dispersion modeling, Robin-type boundary condition, Gaussian plume equation, Green's function method, K-theory.

NOMENCLATURE

α	power-law constant of wind profile, dimensionless
β	power-law constant of vertical eddy diffusivity profile, dimensionless
a	parameter in power-law wind profile, $L^{1-\alpha}t^{-1}$
b	parameter in power-law vertical eddy diffusivity profile, $L^{2-\beta}t^{-1}$
A, B	constants
$C(x, y, z)$	ambient concentration of the contaminant, ML^{-3}
C_c	characteristic concentration, ML^{-3}
$C_u(x, z)$	ambient concentration of the contaminant from infinite line source of unit strength, same units as G_z, $L^{-2}t$
$G_y(x, y)$	crosswind dispersion factor, L^{-1} for point source, dimensionless for line source
$G_z(x_0, z_0; x, z)$	vertical Green's function, $L^{-2}t$
$G_{z,j}$	vertical sub-Green function, $L^{-2}t$
h	characteristic mixing height, L
K_b	vertical eddy diffusion coefficient at $z = z_b$, L^2t^{-1}
K_z	vertical eddy diffusivity, L^2t^{-1}
L	differential operator, defined in equation (1)
L^*	adjoint differential operator, defined in equation (3)
L_S	length of line source, L
Q_p	emission strength of point source, Mt^{-1}
Q_l	emission strength of line source, $ML^{-1}t^{-1}$
S	source term in atmospheric diffusion equation, $ML^{-3}t^{-1}$
U	wind speed, Lt^{-1}
V	general function to replace $G_{z,0}$ in equation (15_1)
v_d	dry deposition velocity, Lt^{-1}
x, y, z	Cartesian coordinates in downwind, crosswind, and vertical directions (positive upwards), respectively, L
x_0, z_0	independent variables, L
x_S, y_S, z_S	location of the point or line source, L
z_b	lower boundary height where deposition flux is measured, L
z_r	reference height where measurements are taken, L
erf	error function
erfc	complementary error function
I	modified Bessel function of the first kind
δ	Dirac delta function
Γ	Gamma function
μ	$= (1-\beta)/(\alpha-\beta+2)$, dimensionless
σ_y	standard deviation (diffusion coefficient), L
ξ	dummy variable in integral.

*Author to whom correspondence should be addressed.

1. INTRODUCTION

Dry deposition is an important removal process for some atmospheric pollutants. The uptake of the pollutants at the earth's surface, either by soil, water, or vegetation, reduces airborne concentration levels at locations far downwind, while potentially increasing exposure levels at nearby locations due to the deposited material. Mathematically modeling the transport of airborne pollutants with dry deposition was first attempted by modifying the Gaussian plume equation to account for this removal process. These modifications have included source depletion models (Chamberlain, 1953; Overcamp, 1976) and surface depletion models (Horst, 1977, 1984). A more theoretical approach to simulating the dispersion–deposition process focused on solving the atmospheric diffusion equation with a radiation boundary condition at the ground (e.g., Seinfeld, 1986). Early analytical solutions (Smith, 1962; Ermak, 1977; Rao, 1981) retained the framework of the Gaussian plume approach, i.e., invariant wind speed and eddies with height. More recent efforts to solve the diffusion equation have utilized height-dependent wind speed and eddy diffusivities (Horst and Slinn, 1984; Koch, 1989; Chrysikopoulos *et al.*, 1992). However, these solutions either contain the unknown in an integral equation or include a hypergeometric function, both of which pose difficulties in computer programming. Those approaches are also restricted to the special case where the source is located at ground level. To the authors' knowledge, no analytical solution has been published which considers a more general case (e.g., near-ground level emissions at an arbitrary source height) and provides a solution that is amenable to modeling applications.

This paper presents a generalized mathematical scheme that solves the atmospheric diffusion equation with arbitrary power-law functions of wind speed and eddy diffusivities. A boundary condition of the Robin type is imposed at the ground to simulate the deposition process. The source can be located anywhere within the region of interest. This scheme builds upon the authors' previous work in which Green's functions were systematically derived as building blocks of the analytical solutions for various homogeneous boundary types. Although relevant results will be summarized where needed, readers interested in the details should refer to the authors' earlier paper (Lin and Hildemann, 1996).

In what follows, an introduction to the powerful Green's function method is presented. Step-by-step procedures for obtaining the new analytical solution based on this solving technique are described, followed by a discussion of the results.

2. ATMOSPHERIC DIFFUSION EQUATION

The three-dimensional atmospheric diffusion equation becomes analytically solvable for the particular case where the wind speed and the lateral eddy diffusivity have the same, but arbitrary, power-law dependence on height (Yeh, 1975). Under this condition, the three-dimensional diffusion equation can then be separated into a pair of two-dimensional (x–z and x–y) diffusion equations. Since the boundary conditions are to be placed in the vertical z direction, with no effect on the diffusion equation in the x–y plane, the two-dimensional equation in the x–z direction will be considered to simplify the discussion. The concentrations obtained hereafter can be multiplied with ease by a cross-wind dispersion factor, G_y, to include lateral diffusion (Lin and Hildemann, 1996; see Table 1 for a summary).

The two-dimensional steady-state diffusion equation for a nonreactive, continuously released contaminant is equivalent to the diffusion equation used for an infinite line source:

$$U(z)\frac{\partial C(x,z)}{\partial x} = \frac{\partial}{\partial z}\left(K_z(z)\frac{\partial C(x,z)}{\partial z}\right) + S(x,z)$$

where x and z are the Cartesian coordinates in the downwind direction and the vertical (positive upwards) direction, respectively, $C(x, z)$ is the ambient concentration of the contaminant, and $S(x, z)$ is the source strength function (mass/vol air-time). Assumptions in the above *atmospheric diffusion equation* include unidirectional wind, gradient turbulent flux (K-theory), and negligible turbulent diffusion (compared to advection) in the wind direction. In order to obtain analytical (exact) solutions, wind speed $U(z)$ and the vertical eddy diffusivity $K_z(z)$ are approximated by the following power-law functions of height:

$$U(z) = U(z_r)\left(\frac{z}{z_r}\right)^{\alpha} = az^{\alpha}, \quad a = \frac{U(z_r)}{z_r^{\alpha}}$$

$$K_z(z) = K_z(z_r)\left(\frac{z}{z_r}\right)^{\beta} = bz^{\beta}, \quad b = \frac{K_z(z_r)}{z_r^{\beta}}$$

where $U(z_r)$ and $K_z(z_r)$ are the measured wind speed and vertical eddy diffusivity at a reference height z_r, and a, b, α, β, are constants that depend on atmospheric stability and surface roughness (Brutsaert and Yeh, 1970a, b; Yeh and Huang, 1975).

3. GREEN'S FUNCTION METHOD

The Green's function concept (Roach, 1970; Greenberg, 1971; Stakgold, 1979) is a powerful tool with which to analyze partial differential equations. Because it is especially useful for boundary value problems with complicated boundary conditions and Dirac delta functions, it merits a brief introduction. The two-dimensional atmospheric diffusion equation will be used as an example for illustration. By defining a differential operator L acting on argument $C(x, z)$, and substituting in the power-law expressions for $U(z)$ and $K_z(z)$, the atmospheric diffusion equation can be recast as follows:

$$L[C(x_0, z_0)] = \frac{\partial}{\partial z_0}\left(bz_0^{\beta}\frac{\partial C(x_0, z_0)}{\partial z_0}\right) - az_0^{\alpha}\frac{\partial C(x_0, z_0)}{\partial x_0} \tag{1}$$

Table 1. Cross-wind dispersion factor G_y for three-dimensional dispersion-deposition modeling, $C(x, y, z) = Q \times C_u(x, z) \times G_y$

Source type	Emission strength Q (units)	Cross-wind dispersion factor G_y (units)
Point source	Q_p(kg/sec)	$G_y = \dfrac{1}{\sqrt{2\pi}\sigma_y(x - x_S)} \exp\left[-\dfrac{(y - y_S)^2}{2\sigma_y^2(x - x_S)}\right] (\mathrm{m}^{-1})$
Finite line source	Q_l(kg/sec-m)	$G_y = \dfrac{1}{2}\left[\mathrm{erf}\left(\dfrac{(L_S/2) - y}{\sqrt{2}\,\sigma_y(x - x_S)}\right) + \mathrm{erf}\left(\dfrac{(L_S/2) + y}{\sqrt{2}\,\sigma_y(x - x_S)}\right)\right] (-)$
Infinite line source	Q_l(kg/sec-m)	$G_y = 1\ (-)$

Note: $C_u(x, z)$: ambient concentration from an infinite line source of unit strength (derived in this paper); erf: error function; x_S, y_S: coordinates of the point source; L_S: length of the line source; $\sigma_y(x)$: standard deviation or turbulent diffusion coefficient in cross-wind direction evaluated at x.

$$L[C(x_0, z_0)] = -S(x_0, z_0) \tag{2}$$

where x_0 and z_0 have been used as independent variables instead of x and z for the purpose of ending up with $C(x, z)$, rather than $C(x_0, z_0)$, in the solution (Greenberg, 1971). Define another (similar) differential operator L^* (known as an adjoint operator in mathematical physics) acting on argument $G_z(x_0, z_0; x, z)$, the Green's function, with the sign of the first-order differential term in L reversed. Assign its value to be a two-dimensional Dirac delta function $\delta(x_0 - x)\delta(z_0 - z)$:

$$L^*[G_z(x_0, z_0; x, z)] = \frac{\partial}{\partial z_0}\left(b z_0^{\beta} \frac{\partial G_z(x_0, z_0; x, z)}{\partial z_0}\right) + a z_0^{\alpha} \frac{\partial G_z(x_0, z_0; x, z)}{\partial x_0} \tag{3}$$

$$L^*[G_z(x_0, z_0; x, z)] = -\delta(x_0 - x)\delta(z_0 - z) \tag{4}$$

where x and z are the "original" independent variables. For the analyses presented in this paper, the domain will be assumed to have no upper bound in the vertical direction, allowing contaminants to diffuse freely with no interference from an inversion layer. Multiplying equation (2) by $G_z(x_0, z_0; x, z)$ and equation (4) by $C(x_0, z_0)$, subtracting one from the other, integrating the result over its domain (x_0 from 0 to x, z_0 from 0 to ∞), and transforming the double integral into a single boundary integral by the divergence theorem, we obtain the unknown $C(x, z)$ without actually solving the differential equation:

$$C(x, z) = -\int_0^{\infty} a z_0^{\alpha} [G_z(x_0, z_0; x, z) C(x_0, z_0)]|_{x_0 = 0} \mathrm{d}z_0 \tag{5_1}$$

$$+ \int_0^x \int_0^{\infty} G_z(x_0, z_0; x, z) S(x_0, z_0) \mathrm{d}z_0 \mathrm{d}x_0 \tag{5_2}$$

$$+ \int_0^x b z_0^{\beta} \left[G_z(x_0, z_0; x, z) \frac{\partial C(x_0, z_0)}{\partial z_0} - C(x_0, z_0) \frac{\partial G_z(x_0, z_0; x, z)}{\partial z_0} \right] \Bigg|_{z_0 = 0}^{z_0 = \infty} \mathrm{d}x_0 \tag{5_3}$$

In this equation, the signs of the outward normal of a differential area from the divergence theorem have been incorporated into the upper and lower limits of the vertical bars, and the causality condition (Yeh and Brutsaert, 1971; Beck *et al.*, 1992) has been applied to remove the upper limit in integral (5_1). In equation (5), the concentration $C(x, z)$ has been expressed in terms of the Green's function $G_z(x_0, z_0; x, z)$. The three integrals, (5_1)–(5_3), represent the pulse contributions from the "initial" condition $x_0 = 0$, the source function $S(x_0, z_0)$, and the boundary conditions, respectively.

The major advantage of this method is that, once the Green's function is known, the solutions of different but similar problems can be obtained by adroit manipulations of these three integrals. For example, a variety of boundary conditions, either homogeneous, nonhomogeneous or nonlinear, can be incorporated into integral (5_3) easily (as will be shown in the following sections). In addition, the equation is written in an integral form—a "smoother" operator for numerical evaluation than a differential operator, with which one would have to deal if one were to directly solve the original problem numerically. This method is most suitable for problems containing a Dirac delta function, because its "shifting" behavior (i.e., $\int f(\xi)\delta(\xi - \rho)\, d\xi = f(\rho)$) allows the integrations to be evaluated immediately. The difficulty, however, is to find the Green's function that satisfies equation (4). The applicability of the method thus rests upon whether such a Green's function exists.

Equations of diffusion, such as the atmospheric diffusion equation, often involve space or time Dirac delta terms where pollution sources are "initiated". There are two ways to stipulate this source information (e.g., Seinfeld, 1986; Deng and Horne, 1993) which will provide a condition on x_0 needed for analytical solution. One approach is to place the source (Dirac delta) at the location $x_0 = x_S = 0$ (Calder, 1961; Runca and Sardei, 1975; Maul, 1977; Demuth, 1978; Tirabassi *et al.*, 1986; Koch, 1989), by which (5_2)

will vanish. The other is to include delta in the source term S (Melli and Runca, 1979; Robson, 1983; Chrysikopoulos *et al.*, 1992), by which integral (5_1) will vanish. The outcome will be the same either way. In other words, only two of the integrals, (5_1) and (5_3), or (5_2) and (5_3), are needed. Though less commonly used in air dispersion modeling, the second approach, formulated as equation (6), will be adopted here for better flexibility and easier interpretation. This leaves us with integrals (5_2) and (5_3) for determining the unknown $C(x, z)$.

$$\frac{S(x_0, z_0)}{Q_l(x_S, z_S)} = \delta(x_0 - x_S)\,\delta(z_0 - z_S). \qquad (6)$$

In equation (6), x_S and z_S can be thought of as coordinates of a line source, where a unit source strength has been assumed for convention. For this case, the concentration obtained later in the paper (written as C_u) should be multiplied by Q_l (kg s^{-1} m^{-1}) to adjust for the line source strength.

The original differential equation, equation (2), requires two boundary conditions (at $z = 0$ and at $z = \infty$ in the present example for an unbounded atmosphere) to obtain a solution, as does the "shifted" problem, equation (4), to determine the Green's function. The upper boundary condition at $z = \infty$ for the unknown C, and also for the Green's function $G_z(x_0, z_0; x, z)$, assumes the disappearance of both pollutant and pollutant flux:

$$C(x_0, z_0) = 0 \text{ and } \frac{\partial C(x_0, z_0)}{\partial z_0} = 0 \text{ at } z_0 = \infty \qquad (7_1)$$

$$G_z(x_0, z_0; x, z) = 0 \text{ and } \frac{\partial G_z(x_0, z_0; x, z)}{\partial z_0} = 0 \text{ at } z_0 = \infty. \qquad (7_2)$$

4. BOUNDARY CONDITIONS OF THE FIRST AND SECOND KINDS

The lower boundary condition of the Green's function is chosen depending on whether the original condition is the first kind (Dirichlet type, $C(x_0, z_0) = 0$ at $z_0 = 0$; total adsorption) or the second kind (Neumann type, $\partial C(x_0, z_0)/\partial z_0 = 0$ at $z_0 = 0$; total reflection) as follows:

Dirichlet:

$$G_z(x_0, z_0 = 0; x, z) = 0 \quad \text{if} \quad C(x_0, z_0 = 0) = 0 \qquad (8_1)$$

Neumann:

$$\left.\frac{\partial G_z(x_0, z_0; x, z)}{\partial z_0}\right|_{z_0 = 0} = 0 \quad \text{if} \quad \left.\frac{\partial C(x_0, z_0)}{\partial z_0}\right|_{z_0 = 0} = 0 \qquad (8_2)$$

In either case, equation (5), which has been reduced to only integrals (5_2) and (5_3) by our decision to put the Dirac delta in the source term S, can be further reduced to integral (5_2) only, because the boundary conditions given in equations (7) and (8) cause the boundary integral (5_3) to vanish. Thus,

$$C_u(x, z) = \int_0^x \int_0^\infty G_z(x_0, z_0; x, z)\,\delta(x_0 - x_S) \times \delta(z_0 - z_S)\,\mathrm{d}z_0\,\mathrm{d}x_0 = G_z(x_S, z_S; x, z)$$

where the shifting behavior of the Dirac delta function has been utilized (i.e., arguments x_0 and z_0 have been replaced by x_S and z_S, respectively), and $G_z(x_0, z_0; x, z)$ is to be determined by partial differential equation (4), the upper boundary condition (7_2), and the lower boundary condition (8_1) or (8_2), depending on the criteria. The Green's functions for these situations, that is, the homogeneous boundary conditions without the inversion layer present, as well as those with inversion effects, have been systematically derived in Lin and Hildemann (1996). These Green's functions can serve as building blocks for other similar but more complicated problems, as will be demonstrated in the next two sections.

5. BOUNDARY CONDITIONS OF THE THIRD KIND

An example of a boundary condition of the third kind is the dry deposition process, which can be mathematically formulated as (Monin, 1959; Calder, 1961):

$$v_d \lim_{z_0 \to z_b} C(x_0, z_0) = \lim_{z_0 \to z_b} K_z(z_0) \frac{\partial C(x_0, z_0)}{\partial z_0} = K_b \lim_{z_0 \to z_b} \frac{\partial C(x_0, z_0)}{\partial z_0} \qquad (9_1)$$

where v_d is the dry deposition velocity, whose sign depends on whether the ground surface is a sink (positive) or a source (negative) (Koch, 1989), z_b is the lower boundary height where deposition flux is measured, and K_b is the vertical eddy diffusivity at $z_0 = z_b$ (i.e., $K_b = bz_b^\xi$). Utilizing the approach of Gillani (1978), equation (9_1) can be nondimensionalized, using C_c as a characteristic concentration, and h as a characteristic mixing height ($v_d h/K_b$ represents the turbulent Sherwood number for vertical surface layer mass transfer):

$$\frac{\partial(C(x_0, z_0)/C_c)}{\partial(z_0/h)} = \frac{v_d h}{K_b} \frac{C(x_0, z_0)}{C_c} \quad \text{as} \quad \frac{z_0}{h} \to 0 \qquad (9_2)$$

With a mixed-type boundary condition such as equation (9_2), the boundary value problem can be solved, numerically or in limited cases analytically, by manipulating equation (5). For example, using the Green's function of the Neumann type ($G_z(\mathbf{ZR}_3)$ in Table 2) as the building block (thus $\partial G_z(x_0, z_0; x, z)/\partial z_0|_{z_0=0} = 0$), and replacing $\partial C(x_0, z_0)/\partial z_0$ in equation (5) by $v_d C(x_0, z_0)/K_b$, yields a nonhomogeneous integral equation of the second kind (Tricomi, 1985; Dettman, 1988; Sobolev, 1989)—an equation where the unknown appears inside an integral and on both

Table 2. Selected Green's functions in an unbounded region[a,b]

Type	Conditions	Green's function
Neumann	$\dfrac{\partial G_z(x_0, z_0; x, z)}{\partial z_0} = 0 \quad \text{at } z_0 = 0$	$G_z(\mathbf{ZR}_3; x_0, z_0; x, z) = \dfrac{(zz_0)^{(1-\beta)/2}}{b(\alpha-\beta+2)(x-x_0)} I_{-\mu}\left[\dfrac{2a(zz_0)^{(\alpha-\beta+2)/2}}{b(\alpha-\beta+2)^2(x-x_0)}\right] \times \exp\left[-\dfrac{a(z^{\alpha-\beta+2}+z_0^{\alpha-\beta+2})}{b(\alpha-\beta+2)^2(x-x_0)}\right]$
Dirichlet	$G_z(x_0, z_0; x, z) = 0 \quad \text{at } z_0 = 0$	$G_z(\mathbf{ZA}_2; x_0, z_0; x, z) = \dfrac{(zz_0)^{(1-\beta)/2}}{b(\alpha-\beta+2)(x-x_0)} I_{\mu}\left[\dfrac{2a(zz_0)^{(\alpha-\beta+2)/2}}{b(\alpha-\beta+2)^2(x-x_0)}\right] \times \exp\left[-\dfrac{a(z^{\alpha-\beta+2}+z_0^{\alpha-\beta+2})}{b(\alpha-\beta+2)^2(x-x_0)}\right]$
Neumann	Limiting case $z_0 = 0$; $\dfrac{\partial G_z(x_0, z_0; x, z)}{\partial z_0} = 0 \quad \text{at } z_0 = 0$	$G_z(\mathbf{ZR}_4; x_0, z_0 = 0; x, z) = \dfrac{\alpha-\beta+2}{a\Gamma\left(\dfrac{\alpha+1}{\alpha-\beta+2}\right)}\left[\dfrac{a}{b(\alpha-\beta+2)^2(x-x_0)}\right]^{(\alpha+1)/(\alpha-\beta+2)} \times \exp\left[-\dfrac{az^{\alpha-\beta+2}}{b(\alpha-\beta+2)^2(x-x_0)}\right]$
	Gaussian plume type ($\alpha = \beta = 0$, $\mu = \frac{1}{2}$)	
Neumann	$\dfrac{\partial G_z(x_0, z_0; x, z)}{\partial z_0} = 0 \quad \text{at } z_0 = 0$	$G_z(\mathbf{ZG}_5; x_0, z_0; x, z) = \dfrac{1}{\sqrt{4\pi ab(x-x_0)}}\left\{\exp\left[-\dfrac{a(z-z_0)^2}{4b(x-x_0)}\right] + \exp\left[-\dfrac{a(z+z_0)^2}{4b(x-x_0)}\right]\right\}$
Dirichlet	$G_z(x_0, z_0; x, z) = 0 \quad \text{at } z_0 = 0$	$G_z(\mathbf{ZG}_6; x_0, z_0; x, z) = \dfrac{1}{\sqrt{4\pi ab(x-x_0)}}\left\{\exp\left[-\dfrac{a(z-z_0)^2}{4b(x-x_0)}\right] - \exp\left[-\dfrac{a(z+z_0)^2}{4b(x-x_0)}\right]\right\}$

[a]For a complete list of the Green's functions, refer to Lin and Hildemann (1996).
[b]Except for $G_z(\mathbf{ZR}_4)$, all the Green's functions are reciprocal (interchangeable) between z_0 and z.

sides of the equation:

$$
\begin{aligned}
C(x, z) &= \int_0^x \int_0^\infty G_z(\mathbf{ZR}_3; x_0, z_0; x, z)\,\delta(x_0 - x_S)\,\delta(z_0 - z_S)\,\mathrm{d}z_0\,\mathrm{d}x_0 \\
&\quad + \int_0^x bz_0^\beta \left[G_z(\mathbf{ZR}_3; x_0, z_0; x, z)\frac{\partial C(x_0, z_0)}{\partial z_0}\right]\Bigg|_{z_0 = z_b \to 0} \mathrm{d}x_0 \\
&= G_z(\mathbf{ZR}_3; x_S, z_S; x, z) + v_d \int_0^x G_z(\mathbf{ZR}_4; x_0, z_0 = 0; x, z)\,[C(x_0, z_0)]|_{z_0 = 0}\,\mathrm{d}x_0.
\end{aligned}
$$

The first integration in the above equation has been carried out by replacing arguments x_0 by x_S and z_0 by z_S, and the building block $G_z(\mathbf{ZR}_3; x_0, z_0; x, z)$ in the second integral has been replaced by its limiting case $G_z(\mathbf{ZR}_4; x_0, z_0 = 0; x, z)$ from Table 2. Notice that since the second integral is to be evaluated at the lower boundary, the sign of the integral should be changed when it is evaluated. Yeh and Brutsaert (1970, 1971) have treated a similar equation of this kind, but in the present case a solution in a mathematically closed form is very difficult to obtain using their approach. A better alternative leading to a closed- form solution is described in the next section.

6. GENERALIZED DEPOSITION SCHEME

The boundary integral (5_3), which is the cause of the troublesome integral equation, can be avoided by imposing a boundary condition analogous to equation (9_2) on $G_z(x_0, z_0; x, z)$:

$$\frac{\partial G_z(x_0, z_0; x, z)}{\partial z_0} = \frac{v_d}{K_b} G_z(x_0, z_0; x, z) \quad \text{at} \quad z_0 = 0. \tag{10}$$

Integral (5_3) is then eliminated as a result of boundary conditions (7), (9_2), and (10). Once the Green's function of this situation is found, the unknown $C(x, z)$ can be obtained immediately by exploiting the shifting behavior of the Dirac delta function. However, the problem has not been solved, as the difficulty is merely switched from solving an integral equation to another equally difficult task—finding the Green's function that satisfies equations (4), (7_2), and (10).

The mathematical scheme used here to accomplish this task is an extended version of the perturbation method described in Greenberg (1971), where a simple Laplace equation with constant coefficients was discussed. Expanding $G_z(x_0, z_0; x, z)$ to be a power-law summation in v_d/K_b:

$$G_z(x_0, z_0; x, z) = G_{z,0} + \left(\frac{v_d}{K_b}\right) G_{z,1} + \left(\frac{v_d}{K_b}\right)^2 G_{z,2} + \left(\frac{v_d}{K_b}\right)^3 G_{z,3} + \cdots + \left(\frac{v_d}{K_b}\right)^j G_{z,j} + \cdots \tag{11}$$

Inserting it into equation (4), and equating the coefficients of the like powers of v_d/K_b on both sides of the equation, yields:

$$\text{for} \quad z_0 > 0 \begin{cases} L^*[G_{z,0}] = -\delta(x_0 - x)\delta(z_0 - z), & (12) \\ L^*[G_{z,j}] = 0 \quad \text{for} \quad j \geqslant 1. & (13) \end{cases}$$

Similarly, inserting the power expansion of $G_z(x_0, z_0; x, z)$ into the boundary condition given in equation (10), and equating like powers of v_d/K_b, yields:

$$\frac{\partial G_{z,0}}{\partial z_0} + \left(\frac{v_d}{K_b}\right)\frac{\partial G_{z,1}}{\partial z_0} + \left(\frac{v_d}{K_b}\right)^2 \frac{\partial G_{z,2}}{\partial z_0} + \left(\frac{v_d}{K_b}\right)^3 \frac{\partial G_{z,3}}{\partial z_0} + \cdots = \left(\frac{v_d}{K_b}\right) G_{z,0} + \left(\frac{v_d}{K_b}\right)^2 G_{z,1} + \left(\frac{v_d}{K_b}\right)^3 G_{z,2} + \cdots \tag{14}$$

$$\text{at} \quad z_0 = 0 \begin{cases} \dfrac{\partial G_{z,0}}{\partial z_0} = 0 & (15_0) \\ \dfrac{\partial G_{z,1}}{\partial z_0} - G_{z,0} = 0 & (15_1) \\ \dfrac{\partial G_{z,2}}{\partial z_0} - G_{z,1} = 0 & (15_2) \\ \dfrac{\partial G_{z,3}}{\partial z_0} - G_{z,2} = 0 & (15_3) \\ \vdots & \\ \vdots & \\ \dfrac{\partial G_{z,j}}{\partial z_0} - G_{z,j-1} = 0. & (15_j) \\ \vdots & \end{cases}$$

The first sub-Green function $G_{z,0}$, satisfying equation (12) (or L^* operator) and boundary condition (15_0), is the Neumann-type Green's function $G_z(\mathbf{ZR}_3; x_0, z_0; x, z)$ (solved in Lin and Hildemann, 1996):

$$G_{z,0}(x_0, z_0; x, z) = G_z(\mathbf{ZR}_3; x_0, z_0; x, z). \tag{16}$$

The other sub-Green functions $G_{z,j}$ will be determined from equation (13) and boundary conditions (15_j) ($j \geqslant 1$). Applying the L^* operator to equation (15_j) for $j \geqslant 2$, and using the fact that all $G_{z,j}$ ($j \geqslant 1$) satisfy L^* for $z_0 > 0$ (i.e., equation (13)):

$$L^*[\text{equation } (15_j)] = L^*\left[\frac{\partial G_{z,j}}{\partial z_0} - G_{z,j-1}\right]$$

$$= L^*\left[\frac{\partial G_{z,j}}{\partial z_0}\right] - L^*[G_{z,j-1}]$$

$$= \frac{\partial}{\partial z_0}(L^*[G_{z,j}]) - L^*[G_{z,j-1}] = 0 \quad \text{for} \quad j \geqslant 2$$

we find that, although written to be valid only on the boundary $z_0 = 0$, due to the fact that L^* is satisfied for $z_0 > 0$, each of the equations (15_j), except $j = 0$ and 1, is in fact valid in the entire region (i.e., for $z_0 \geqslant 0$) as a result of Green's uniqueness theorem. However, equation (15_1) applies *only* on the boundary because $G_{z,0}$ in it also simultaneously satisfies equation (15_0). The key to this method is to find a less restrictive (or more general) function (denoted as $V(x_0, z_0; x, z)$) to replace $G_{z,0}$ in equation (15_1) so that equation (15_1) can also apply to the entire region, like the other (15_j) equations. From Yeh and Brutsaert (1971) and Lin and Hildemann (1996), the general solution of L^* without considering any boundary condition is the linear combination of the Neumann-type Green's function ($\mathbf{ZR}_3$) and the Dirichlet-type

Green's function ($\mathbf{ZA}_2$) (see Table 2):

$$V(x_0, z_0; x, z) = A \times G_z(\mathbf{ZR}_3; x_0, z_0; x, z) + B \times G_z(\mathbf{ZA}_2; x_0, z_0; x, z) = A \times G_{z,0}(x_0, z_0; x, z) + B \times G_z(\mathbf{ZA}_2; x_0, z_0; x, z)$$

where A and B are constants to be determined. In determining the two constants, it is imperative that the properties and roles of the substituted function $G_{z,0}$ in equations (15_0) and (15_1) be retained in the substituting function $V(x_0, z_0; x, z)$. First, $V(x_0, z_0 = 0; x, z)$ must equal $G_{z,0}(x_0, z_0 = 0; x, z)$ at the boundary, which requires A to be 1. In addition, since equation (15_1) is to be extended into the entire region, V must be positive and its derivative $\partial V/\partial z_0$ negative (and importantly, nonzero) at all values of z, which leaves as the only possible choice $B = -1$ (see Fig. 2). Thus,

$$V(x_0, z_0; x, z) = G_z(\mathbf{ZR}_3; x_0, z_0; x, z) - G_z(\mathbf{ZA}_2; x_0, z_0; x, z). \tag{17}$$

In brief, $G_{z,0}$ is known (equation (16)), its role in equation (15_1) is replaced by a derived function V (equation (17)), and most importantly, in so doing, boundary conditions (15_j) for $j \geqslant 1$ can all be extended throughout the entire domain (Greenberg, 1971). Integrating equations (15_j) for $j \geqslant 1$ successively:

$$G_{z,1} = \int V \, dz_0 \tag{18_1}$$

$$G_{z,2} = \int G_{z,1} \, dz_0 = \iint V \, dz_0 \, dz_0 \tag{18_2}$$

$$G_{z,3} = \int G_{z,2} \, dz_0 = \iiint V \, dz_0 \, dz_0 \, dz_0 \tag{18_3}$$

$$\vdots$$

$$G_{z,j} = \int G_{z,j-1} \, dz_0 = \underbrace{\iint \cdots \int}_{j \text{ items}} V \underbrace{dz_0 \, dz_0 \, dz_0 \cdots dz_0}_{j \text{ items}} \tag{18_j}$$

Substituting equation (18) back into the power expansion of G_z, and differentiating it with respect to z_0, yields equation (19):

$$G_z(x_0, z_0; x, z) = G_{z,0} + \left(\frac{v_d}{K_b}\right) \int V \, dz_0 + \left(\frac{v_d}{K_b}\right)^2 \iint V \, dz_0 dz_0 + \left(\frac{v_d}{K_b}\right)^3 \iiint V \, dz_0 \, dz_0 \, dz_0 + \cdots + \left(\frac{v_d}{K_b}\right)^j \underbrace{\iiint \cdots \int}_{j \text{ items}} V \underbrace{dz_0 \, dz_0 \, dz_0 \cdots dz_0}_{j \text{ items}} + \cdots$$

$$\frac{\partial G_z}{\partial z_0} = \frac{\partial G_{z,0}}{\partial z_0} + \left(\frac{v_d}{K_b}\right) V + \left(\frac{v_d}{K_b}\right)^2 \int V \, dz_0 + \left(\frac{v_d}{K_b}\right)^3 \iint V \, dz_0 + \cdots + \left(\frac{v_d}{K_b}\right)^j \underbrace{\iint \cdots \int}_{j-1 \text{ items}} V \underbrace{dz_0 \, dz_0 \cdots dz_0}_{j-1 \text{ items}} + \cdots$$
$$= \frac{\partial G_{z0}}{\partial z_0} + \frac{v_d}{K_b} V + \frac{v_d}{K_b}(G_z - G_{z,0}). \tag{19}$$

Rearranging equation (19) yields a first-order ordinary differential equation (20), whose right hand side is made up of known functions $G_{z,0}$ and V. The solution can be readily obtained using the integrating factor ($e^{-(v_d/K_b)z_0}$ in this case) shown in (21), where the integrating constant is constrained to be zero at infinity, and integration by parts has been used in the second step:

$$\frac{\partial G_z}{\partial z_0} - \frac{v_d}{K_b} G_z = \frac{\partial G_{z,0}}{\partial z_0} + \frac{v_d}{K_b} V - \frac{v_d}{K_b} G_{z,0} \tag{20}$$

$$G_z(x_0, z_0; x, z)$$
$$= e^{(v_d/K_b)z_0} \int_{\infty}^{z_0} \left(\frac{\partial G_{z,0}}{\partial \xi} + \frac{v_d}{K_b} V - \frac{v_d}{K_b} G_{z,0} \right) e^{-(v_d/K_b)\xi} \, d\xi$$
$$= e^{(v_d/K_b)z_0} \left\{ G_{z,0} e^{-(v_d/K_b)\xi} \Big|_{\infty}^{z_0} + \frac{v_d}{K_b} \int_{\infty}^{z_0} G_{z,0} e^{-(v_d/K_b)\xi} \, d\xi + \int_{\infty}^{z_0} \left(\frac{v_d}{K_b} V - \frac{v_d}{K_b} G_{z,0} \right) e^{-(v_d/K_b)\xi} \, d\xi \right\}$$
$$= G_{z,0} + \frac{v_d}{K_b} \exp\left(\frac{v_d}{K_b} z_0\right) \int_{\infty}^{z_0} V \exp\left(-\frac{v_d}{K_b}\xi\right) d\xi$$
$$= G_{z,0}(x_0, z_0; x, z) - \frac{v_d}{K_b} \exp\left(\frac{v_d}{K_b} z_0\right) \times \int_{z_0}^{\infty} V(x_0, \xi; x, z) \exp\left(-\frac{v_d}{K_b}\xi\right) d\xi. \tag{21}$$

Substituting equation (21) into integral (5_2), and invoking the shifting behavior of the Dirac delta function (x_0 and z_0 are replaced by x_S and z_S, respectively), gives the final closed-form solution equation (22), in which the known functions $G_{z,0}$ and V have been substituted in from equations (16) and (17).

$$C_u(x,z) = G_z(\mathbf{ZR}_3; x_S, z_S; x, z) - \frac{v_d}{K_b}\exp\left(\frac{v_d}{K_b}z_S\right)\int_{z_S}^{\infty}[(G_z(\mathbf{ZR}_3; x_S, \xi; x, z) - G_z(\mathbf{ZA}_2; x_S, \xi; x, z)]\exp\left(-\frac{v_d}{K_b}\xi\right)d\xi \tag{22}$$

where

$$G_z(\mathbf{ZR}_3; x_0, z_0; x, z) = \frac{(zz_0)^{(1-\beta)/2}}{b(\alpha-\beta+2)(x-x_0)}I_{-\mu}\left[\frac{2a(zz_0)^{(\alpha-\beta+2)/2}}{b(\alpha-\beta+2)^2(x-x_0)}\right]\exp\left[-\frac{a(z^{\alpha-\beta+2}+z_0^{\alpha-\beta+2})}{b(\alpha-\beta+2)^2(x-x_0)}\right]$$

$$G_z(\mathbf{ZA}_2; x_0, z_0; x, z) = \frac{(zz_0)^{(1-\beta)/2}}{b(\alpha-\beta+2)(x-x_0)}I_{\mu}\left[\frac{2a(zz_0)^{(\alpha-\beta+2)/2}}{b(\alpha-\beta+2)^2(x-x_0)}\right]\exp\left[-\frac{a(z^{\alpha-\beta+2}+z_0^{\alpha-\beta+2})}{b(\alpha-\beta+2)^2(x-x_0)}\right]$$

with $\mu = \dfrac{1-\beta}{\alpha-\beta+2}$.

Attention should be paid to the evolution of the Green's function arguments x_0 and z_0 in the derivation. Readers endeavoring to use equation (22) should take special care in modifying the arguments to match those in that equation (i.e., $x_0 \to x_S$; $z_0 \to z_S$ or ξ).

This new solution possesses several advantages over other solutions derived in the past for a ground level source (Horst and Slinn, 1984; Koch, 1989; Chrysikopoulos *et al.*, 1992). It can be applied to a more general situation, such as arbitrary source height, while still maintaining programming and computing feasibility. The only function involved is the modified Bessel function of the first kind I, which is available as a subroutine in almost all programming languages or commercial computing software packages. The integral in the solution can also be evaluated handily using Simpson's rule or other higher order numerical schemes. Note that for zero deposition ($v_d = 0$), the solution reduces to the total reflection case $G_z(\mathbf{ZR}_3; x_S, z_S; x, z)$, as expected. Moreover, it can be shown to reduce to other solutions that have been previously derived for a simplified case, the Gaussian deposition plume.

7. GAUSSIAN DEPOSITION PLUME

The Gaussian plume equation is a special case where both wind speed and eddy diffusivity are invariant with height (i.e., $\alpha = \beta = 0$, $\mu = 1/2$). The eddy diffusion coefficient K_b in equation (10) is therefore equivalent to the constant eddy diffusivity K_z (or b in present notation) of the Gaussian plume. Using identities for a modified Bessel function of order $\pm 1/2$, kernels $(\mathbf{ZR}_3)$ and $(\mathbf{ZA}_2)$ reduce to $(\mathbf{ZG}_5)$ and $(\mathbf{ZG}_6)$, respectively (Lin and Hildemann, 1996), which are listed in Table 2. By substituting in the two simplified kernels, and replacing v_d/K_b by v_d/b, equation (22) becomes:

$$C_u(x,z) = G_z(\mathbf{ZG}_5; x_S, z_S; x, z) - \frac{v_d}{b}\exp\left(\frac{v_d}{b}z_S\right)\int_{z_S}^{\infty}[G_z(\mathbf{ZG}_5; x_S, \xi; x, z) - G_z(\mathbf{ZG}_6; x_S, \xi; x, z)]\exp\left(-\frac{v_d}{b}\xi\right)d\xi \tag{23}$$

where $(\mathbf{ZG}_5)$ and $(\mathbf{ZG}_6)$ are the Green's functions of the Gaussian plume for the total reflection and total adsorption cases. The subtraction inside the integral can be simplified:

$$C_u(x,z) = G_z(\mathbf{ZG}_5; x_S, z_S; x, z) - \frac{v_d}{b}\exp\left(\frac{v_d}{b}z_S\right) \times \int_{z_S}^{\infty}\left\{\frac{2}{\sqrt{4\pi ab(x-x_S)}}\exp\left[-\frac{a(z+\xi)^2}{4b(x-x_S)}\right]\right\} \times \exp\left(-\frac{v_d}{b}\xi\right)d\xi. \tag{24}$$

This can be shown to be the same equation as appears in the classic text in mathematical physics by Sommerfeld (1949), by changing the space variables x and z to t and x, respectively. The integration in equation (24) can be carried out explicitly and expressed as a complementary error function erfc:

$$C_u(x,z) = \frac{1}{\sqrt{4\pi ab(x-x_S)}}\left\{\exp\left[-\frac{a(z-z_S)^2}{4b(x-x_S)}\right]\exp\left[-\frac{a(z+z_S)^2}{4b(x-x_S)}\right]\right\} - \frac{v_d}{ab}\exp\left[\frac{v_d(z+z_S)}{b}+\frac{v_d^2(x-x_S)}{ab}\right]\operatorname{erfc}\left[\frac{z+z_S+2[v_d(x-x_S)/a]}{2\sqrt{b(x-x_S)/a}}\right]. \tag{25}$$

Equation (25) is the solution derived in the past for the Gaussian deposition plume (e.g., Smith, 1962; Scriven and Fisher, 1975; Ermak, 1977; Rao, 1981).

The two terms in equation (25) are both well-known in heat conduction as well, if the space variable x is changed to the time variable t. The first term is widely recognized, and the second term also appears frequently in the literature (e.g., Özisik, 1968; Crank, 1975; Seinfeld, 1986; Beck *et al.*, 1992).

8. DISCUSSION

In Fig. 1, the asymptotic case of the analytical solution, equation (22), with $\alpha = \beta = 0$ and $\mu = 1/2$ (i.e., Gaussian plume type) is compared with another form of solution expressed as a complementary error function (equation (25)). All parameters are identical with K_b assigned to be the same as b (reason stated before). The command *quad*8, a numerical integration method using the adaptive recursive Newton–Cotes eight-panel rule in *MATLAB*, was used to evaluate the integral in equation (22). The four continuous lines are results obtained from equation (25) for various deposition velocities, while the stars are results calculated from equation (22). The identical "matching" of the two solutions in different forms for each of the deposition velocities validates the accuracy of both the derived solutions and the associated computer codes.

The physics represented by equation (22) can be explained as follows. The first term represents concentration contributions from a totally reflecting plume. The second term with a minus sign represents the modification to the first term (total reflection) as a result of groundlevel depletion via deposition. The higher the deposition velocity v_d, the greater the modification. The integrand in the second term contains a difference (subtraction) between the Green's function for total reflection and that for total adsorption. Figure 2 shows the typical profiles of these two Green's functions at two distances downwind. Because the magnitude of this difference is greatest near ground level and reduces upwards, the influence of the second term is expected to be most pronounced at low elevations. This is further illustrated by Fig. 3, where the integrand of equation (22), dummy variable ξ, and elevation z are plotted as a contour surface. The area (shaded) under a given surface curve is equal to the integral of the second term at a fixed height. A smaller area, and thus a smaller modification, is seen for a higher elevation z. In other words, the shape of the concentration profile will resemble that for a totally reflecting plume at sufficiently high elevations. The effect of the source height z_S can be deduced by its role in the lower integration limit. A plume emitted from a source located at a higher elevation will not be perturbed to a large extent by the ground at small distances downwind, as indicated by the smaller shaded area.

Modeling applications of equation (22) can be demonstrated by considering an elevated infinite line source located at ($x_S = 10$ m, $z_S = 50$ m). Pollutants are assumed to disperse freely in an unbounded atmosphere ($H \to \infty$). Meteorological input parameters are arbitrarily chosen as: $\alpha = 0.29$, $\beta = 0.45$, $a = 1.5(\mathrm{m\,s^{-1}})$ $(\mathrm{m}^{-0.29})$, $b = 5.0(\mathrm{m^2\,s^{-1}})$ $(\mathrm{m}^{-0.45})$, $K_b = 5.0(\mathrm{m^2\,s^{-1}})$. Figure 4 shows the vertical concentration profiles of various deposition velocities at two distances downwind: $x = 150$ m (near-source field) and $x = 500$ m (far-away-from-source field). At $x = 150$ m, the plume has reached the ground and the vertical profile has begun to alter, due to surface depletion. The extent of the profile alteration depends on the magnitude of dry deposition. A pollutant with

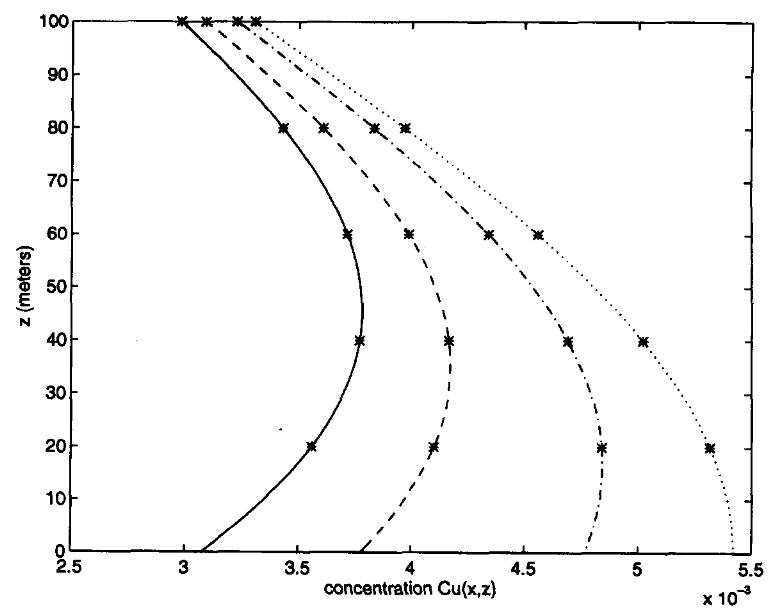

Fig. 1. Comparison of two solutions of Gaussian deposition equation ($\alpha = \beta = 0$, $\mu = \frac{1}{2}$) for four different deposition velocities (0, 0.01, 0.03, and 0.05 m s^{-1}): continuous lines—equation (25); star points—equation (22).

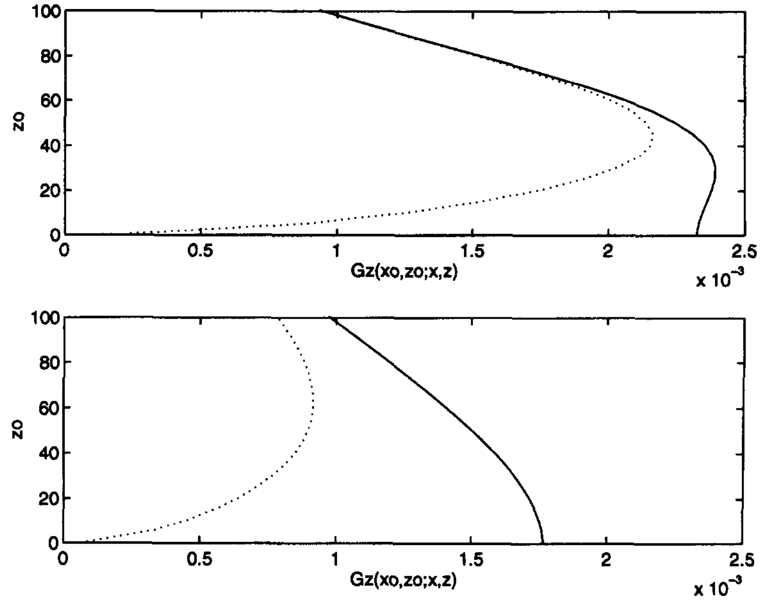

Fig. 2. Typical profiles (in vertical direction z_0) of the Green's functions for total reflection $G_z(\mathbf{ZR}_3)$ (solid line) and total adsorption $G_z(\mathbf{ZA}_2)$ (dotted line) at two distances downwind: (a) short distance; (b) long distance. Notice that the "effective source height" z is at 50 m in this case.

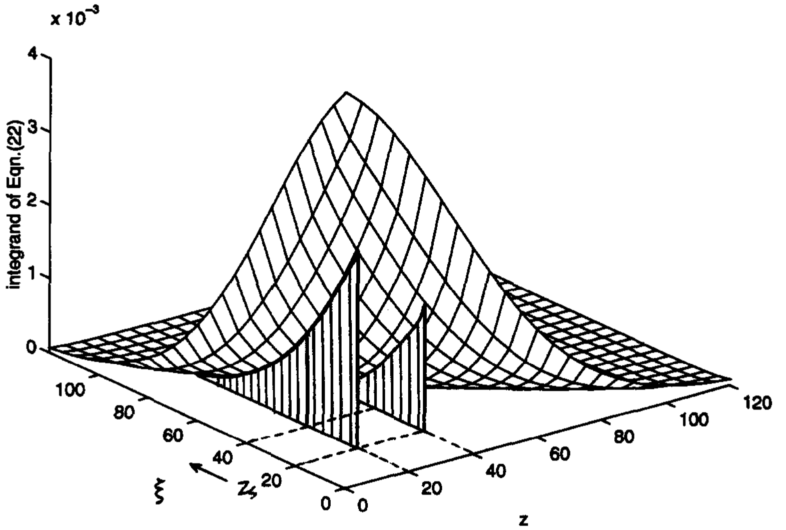

Fig. 3. A typical contour surface of the integrand of equation (22), dummy variable ξ, and elevation z. Shaded area under a surface curve is equivalent to the integral of equation (22) at a fixed height.

zero deposition velocity (dashed line) is totally reflected back into the atmosphere (as seen from the zero-gradient, nonbending profile near the ground). By contrast, a pollutant with the highest deposition velocity (solid line) undergoes substantial depositional losses, thereby reducing the airborne concentrations. At a short distance downwind (Fig. 4a), emissions at higher elevations have not yet experienced the surface effect. Therefore, they maintain the same shape regardless of the magnitude of the deposition velocity. Only when pollutants have traveled sufficiently far downwind (Fig. 4b) is the effect of surface depletion seen at high elevations. Also, for contaminants with nonnegligible deposition velocities, the vertical concentration maximum remains elevated above the earth's surface (Ermak, 1977). Though not easy to measure in the field (Smith, 1962), this positive concentration gradient near the ground has been confirmed by experiments (Berkowicz and Prahm, 1978), and constitutes an important mechanism that most analytical dispersion models (other than the Gaussian plume model) are unable to simulate.

Figure 5 shows the variations of downwind "breathing level" ($z = 2$ m) concentrations with dry

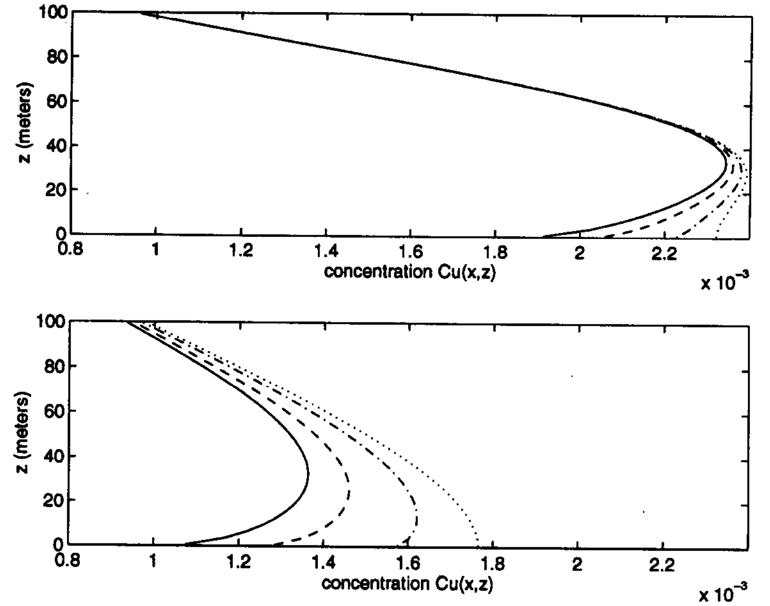

Fig. 4. Variation of vertical concentration profiles with dry deposition velocities (dotted line — $v_d = 0\,\mathrm{m\,s^{-1}}$; dot-dashed line — $v_d = 0.01\,\mathrm{m\,s^{-1}}$; dashed line — $v_d = 0.03\,\mathrm{m\,s^{-1}}$; solid line — $v_d = 0.05\,\mathrm{m\,s^{-1}}$): (a) $x = 150$ m (near-source field); (b) $x = 500$ m (far-away-from-source field).

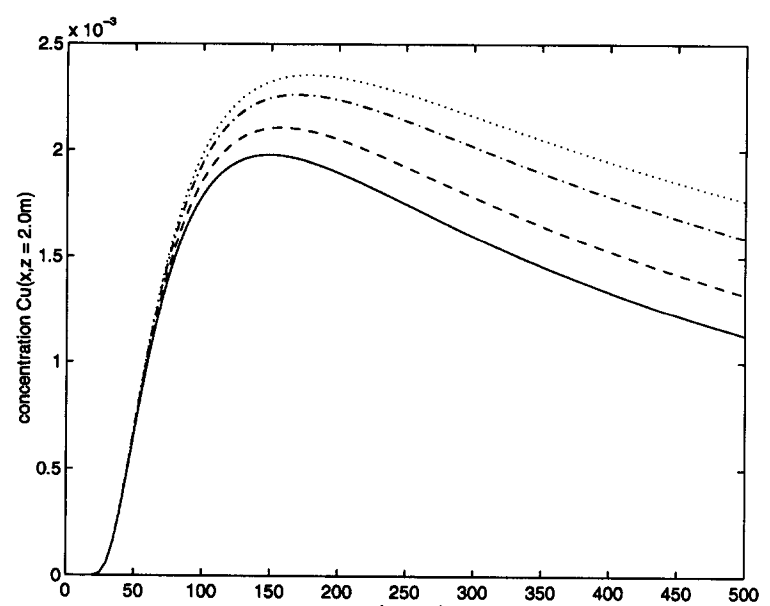

Fig. 5. Variation of downwind breathing level concentration ($z = 2$m) with dry deposition velocities (dotted line — $v_d = 0\,\mathrm{m\,s^{-1}}$; dot-dashed line — $v_d = 0.01\,\mathrm{m\,s^{-1}}$; dashed line — $v_d = 0.03\,\mathrm{m\,s^{-1}}$; solid line — $v_d = 0.05\,\mathrm{m\,s^{-1}}$). Notice that $C_u(x, z) = 0$ for $x < x_S$ in accordance with the causality condition.

deposition velocities. Airborne concentrations are highest for a contaminant that is totally reflected from the ground (dashed line). Increasing the dry deposition reduces the extent of human exposure. The location at which the peak concentration occurs moves closer to the source as depletion becomes greater, a result consistent with predictions obtained with the Gaussian plume solution (Ermak, 1977).

9. SUMMARY

An analytical solution is derived that solves the atmospheric diffusion equation with height-dependent wind speed and eddy diffusivities, and with a Robin-type boundary condition at the ground. The solution consists of two relatively simple Green's functions expressed in an integral form that is amenable to computer programming for modeling applications. In the greatly simplified case where wind speed and eddy diffusivities are invariant with height, the solution reduces to that derived in the past for the Gaussian deposition plume. Examining the physical behavior represented by the new solution confirms that the ground effect is most profound for pollutants dispersing near ground and for emissions from lower elevation sources.

The new solution is in fact a result of a generalized mathematical scheme to analytically solve the atmospheric diffusion equation with dry deposition. The scheme, demonstrated in two dimensions, can be extended by combining a crosswind dispersion factor for three-dimensional dispersion–deposition modeling. Since there is no restriction imposed on the source height, it is very useful for modeling the transport of pollutants released from arbitrary elevations. With slight modifications and appropriate manipulations of other Green's functions, this generalized scheme can be extended further, accounting for inversion effects or leading to other useful solutions.

Acknowledgements—Funding for this study was provided by the U.S. Department of Defense through the U.S. Environmental Protection Agency–supported Western Region Hazardous Substance Research Center, under agreement R-819751, and by the National Science Foundation through its Presidential Young Investigator Program (Grant No. BCS-9157905). The contents of this publication do not necessarily represent the views of these organizations. The authors thank Professors Mark Z. Jacobson and Paul V. Roberts for reviewing and critiquing this paper. Critical comments by the anonymous reviewers were also greatly appreciated.

REFERENCES

Beck J. V., Cole K. D., Haji-Sheikh A. and Litkouhi B. (1992) *Heat Conduction Using Green's Functions*, pp. 41 and 87. Hemisphere Publishing Corp., Washington, District of Columbia.

Berkowicz R. and Prahm L. P. (1978) Pseudospectral simulation of dry deposition from a point source. *Atmospheric Environment* **12,** 379–387.

Brutsaert W. and Yeh G. T. (1970a) Implications of a type of empirical evaporation formula for lakes and pans. *Water Resour. Res.* **6,** 1202–1208.

Brutsaert W. and Yeh G. T. (1970b) A power wind law for turbulent transfer computations. *Water Resour. Res.* **6,** 1387–1391.

Calder K. L. (1961) Atmospheric diffusion of particulate material, considered as a boundary value problem. *J. Met.* **18,** 413–416.

Chamberlain A. C. (1953) Aspects of travel and deposition of aerosol and vapor clouds. A. E. R. E. Report HP/R-1261, Atomic Energy Research Establishment, Harwell, Berkshire, England.

Chrysikopoulos C. V., Hildemann L. M. and Roberts P. V. (1992) A three-dimensional atmospheric dispersion-deposition model for emissions from a ground-level area source. *Atmospheric Environment* **26A,** 747–757.

Crank J. (1975) *The Mathematics of Diffusion*, 2nd Edn., p. 36. Oxford University Press, New York.

Demuth C. (1978) A contribution to the analytical steady solution of the diffusion equation for line sources. *Atmospheric Environment* **12,** 1255–1258.

Deng X. and Horne R. N. (1993) Well test analysis of heterogeneous reservoirs. In 68th Ann. Technical Conf. and Exhibition of the Society of Petroleum Engineers, Houston, Texas, 3–6 October 1993.

Dettman J. W. (1988) *Mathematical Methods in Physics and Engineering*, p. 268. Dover Publications, New York.

Ermak D. L. (1977) An analytical model for air pollutant transport and deposition from a point source. *Atmospheric Environment* **11,** 231–237.

Gillani N. V. (1978) Project MISTT: mesoscale plume modeling of the dispersion, transformation and ground removal of SO_2. *Atmospheric Environment* **12,** 569–588.

Greenberg M. D. (1971) *Application of Green's Functions in Science and Engineering*. Prentice-Hall, Englewood Cliffs, New Jersey.

Horst T. W. (1977) A surface depletion model for deposition from a Gaussian plume. *Atmospheric Environment* **11,** 41–46.

Horst T. W. (1984) The modification of plume models to account for dry deposition. *Boundary-Layer Met.* **30,** 413–430.

Horst T. W. and Slinn W. G. N. (1984) Estimates for pollution profiles above finite area-sources. *Atmospheric Environment* **18,** 1339–1346.

Koch W. (1989) A solution of the two-dimensional atmospheric diffusion equation with height-dependent diffusion coefficient including ground level absorption. *Atmospheric Environment* **23,** 1729–1732.

Lin J. S. and Hildemann L. M. (1996) Analytical solutions of the atmospheric diffusion equation with multiple sources and height-dependent wind speed and eddy diffusivities. *Atmospheric Environment* **30,** 239–254.

Maul P. R. (1977) The mathematical modeling of the mesoscale transport of gaseous pollutants. *Atmospheric Environment* **11,** 1191–1195.

Melli P. and Runca E. (1979) Gaussian plume model parameters for ground-level and elevated sources derived from the atmospheric diffusion equation in a neutral case. *J. appl. Met.* **18,** 1216–1221.

Monin A. S. (1959) On the boundary condition on the earth surface for diffusing pollution, in *Advances in Geophysics*, Vol. 6, pp. 435–436. Academic Press, New York.

Overcamp T. J. (1976) A general Gaussian diffusion-deposition model for elevated point sources. *J. appl. Met.* **15,** 1167–1171.

Özisik M. N. (1968) *Boundary Value Problems of Heat Conduction*, p. 117. Dover Publications, New York.

Rao K. S. (1981) Analytical solutions of a gradient-transfer model for plume deposition and sedimentation, NOAA Technical Memorandum ERL ARL-109.

Roach G. F. (1970) *Green's Functions: Introductory Theory with Applications*. Van Nostrand Reinhold, New York.

Robson R. E. (1983) On the theory of plume trapping by an elevated inversion. *Atmospheric Environment* **17,** 1923–1930.

Runca E. and Sardei F. (1975) Numerical treatment of time dependent advection and diffusion of air pollutants. *Atmospheric Environment* **9,** 69–80.

Scriven R. A. and Fisher B. E. A. (1975) The long range transport of airborne material and its removal by deposition and washout—II. The effect of turbulent diffusion. *Atmospheric Environment* **9,** 59–68.

Seinfeld J. H. (1986) *Atmospheric Chemistry and Physics of Air Pollution*, Chap. 13, pp. 258 and 543. Wiley, New York.

Smith F. B. (1962) The problem of deposition in atmospheric diffusion of particulate matter. *J. atmos. Sci.* **19,** 429–434.

Sobolev S. L. (1989) *Partial Differential Equations of Mathematical Physics*, translated from the Russian edition by E. R. Dawson; English translation edited by T. A. A. Broadbent, p. 237. Dover Publications, New York.

Sommerfeld A. (1949) *Partial Differential Equations in Physics*, p. 66. Academic Press, New York.

Stakgold I. (1979) *Green's Functions and Boundary Value Problems*, Wiley, New York.

Tirabassi T., Tagliazacca M. and Zannetti P. (1986) KAPPA-G, a non-Gaussian plume dispersion model: description and evaluation against tracer measurements. *J. Air Pollut. Control Ass.* **36,** 592–596.

Tricomi F. G. (1985) *Integral Equations*, p. 3. Dover Publications, New York.

Yeh G. T. (1975) Green's function of a diffusion equation. *Geophys. Res. Lett.* **2,** 293–296.

Yeh G. T. and Brutsaert W. (1970) Perturbation solution of an equation of atmospheric turbulent diffusion. *J. geophys. Res.* **75,** 5173–5178.

Yeh G. T. and Brutsaert W. (1971) A solution for simultaneous turbulent heat and vapor transfer between a water surface and the atmosphere. *Boundary-Layer Met.* **2,** 64–82.

Yeh G. T. and Huang C. H. (1975) Three-dimensional air pollution modeling in the lower atmosphere. *Boundary-Layer Met.* **9,** 381–390.