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ANALYTICAL SOLUTIONS OF THE ATMOSPHERIC DIFFUSION EQUATION WITH MULTIPLE SOURCES AND HEIGHT-DEPENDENT WIND SPEED AND EDDY DIFFUSIVITIES

JIN-SHENG LIN and LYNN M. HILDEMANN*
Department of Civil Engineering, Stanford University, Stanford, CA 94305-4020, U.S.A.

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Abstract—Three-dimensional analytical solutions of the atmospheric diffusion equation with multiple sources and height-dependent wind speed and eddy diffusivities are derived in a systematic fashion. For homogeneous Neumann (total reflection), Dirichlet (total adsorption), or mixed boundary conditions, the solutions for a single source are comprised of three components: a source strength, a crosswind dispersion factor, and a vertical dispersion factor. The two dispersion factors together constitute a Green's function—the concentration response due to a unit disturbance (source). When the general point source Green's functions are derived for a bounded domain (inversion effect) with various boundary conditions and arbitrary power-law profiles for wind speed and eddy diffusivities, previously published equations are found to be simplified versions of this more general case. A methodology based on the superposition of Green's functions is proposed, which enables the estimation of ambient concentrations not only from a single source, but also from multiple point, line, or area releases.

Key word index: Dispersion models, Gaussian plume equation, Green's function, inversion layer, *K*-theory, line source.

NOTATION

α	power-law constant of wind profile, dimensionless
β	power-law constant of vertical eddy diffusivity profile, dimensionless
γ	power-law constant of lateral eddy diffusivity profile, dimensionless
a	parameter in power-law wind profile, $L^{1-\alpha}t^{-1}$
b	parameter in power-law vertical eddy diffusivity profile, $L^{2-\beta}t^{-1}$
B	length of line source, L
$C(x, y, z)$	ambient concentration of the contaminant, ML^{-3}
C_1, C_2, C_3, C_4, C_5	constants
$f(x)$	integrable function in lateral eddy diffusivity profile, $L^{2-\alpha}t^{-1}$
$G_y(x, y; x_S^i, y_S^i)$	crosswind sub-Green function, L^{-1} for point source, dimensionless for line source
$G_z(x, z; x_S^i, z_S^i)$	vertical sub-Green function, $L^{-2}t$
$\hat{G}_y(x_0, p; x, y)$	Fourier transformation of $G_y(x_0, y_0; x, y)$ with respect to y_0
H	inversion height, L
i	superscript, index for the *i*th source, dimensionless
j	subscript, index for eigenvalues, dimensionless
k_1, k_2, k_3	constants
K_y	lateral eddy diffusivity, L^2t^{-1}
K_z	vertical eddy diffusivity, L^2t^{-1}
m	transformed variable from $M(z_0)$
$M(z_0)$	function used to separate variables
$N(x_0)$	function used to separate variables
n	number of total emission sources, dimensionless
p	Fourier variable
Q_p	emission strength of point source, Mt^{-1}
Q_ℓ	emission strength of line source, $ML^{-1}t^{-1}$
Q_a	emission strength of area source, $ML^{-2}t^{-1}$
S	source term in atmospheric diffusion equation
t	transformed variable from z_0
U	wind speed, Lt^{-1}
W_0	constant defined in equation (A22)
W_j	constant defined in equation (A24)
x, y, z	Cartesian coordinates in downwind, crosswind, and vertical directions (positive upwards), respectively, L
x_0, y_0, z_0	independent variables, L
x_S^i, y_S^i, z_S^i	location of the *i*th point source, L
$x_{Sl}^i, y_{Sl}^i, z_{Sl}^i$	location of the *i*th line source, L
x_c	some location in the Cartesian coordinate system, L
X	$= \int_{x_c}^{x} f(\tau)\mathrm{d}\tau$, $L^{3-\alpha}t^{-1}$
X_S	$= \int_{x_c}^{x_S^i} f(\tau)\mathrm{d}\tau$, $L^{3-\alpha}t^{-1}$
z_r	reference height where measurements are taken, L
erf	error function

* Corresponding author.

I	modified Bessel function of the first kind
J	Bessel function of the first kind
δ	Dirac delta function
Γ	gamma function
η	argument of a function
κ	separation constant
λ	eigenvalue, dimensionless
μ	$= (1-\beta)/(\alpha-\beta+2)$, dimensionless
ω	$= 2\sqrt{(a/b)}\kappa/(\alpha-\beta+2)$
ζ, ξ	variables in equation (A31)
σ_y	standard deviation (diffusion coefficient), L
$\sigma_y{}^2$	mean square particle displacement, L^2
τ	dummy variable in integral
$d\lvert\mathbf{q}\rvert$	differential source length. L

1. INTRODUCTION

The atmospheric diffusion equation (e.g., Seinfeld, 1986) has long been used to describe the transport of airborne pollutants in a turbulent atmosphere. Air dispersion models based on its analytical solutions possess several advantages over numerical models, because all the influencing parameters are explicitly expressed in a mathematically closed form. The effect of the individual parameters on the model results can therefore be easily investigated (Nieuwstadt, 1980). Analytical solutions are also useful for examining the accuracy and performance of the numerical models (Runca and Sardei, 1975; Liu and Seinfeld, 1975; Runca, 1982). Thorough studies of the analytical solutions allow valuable insights to be gained regarding the behavior of a system.

An analytical solution that has received much attention and has been studied extensively is the Gaussian plume equation, which assumes that wind speed and turbulent eddies are invariant with height. Despite its popularity, studies have shown that its applicability is quite limited. For example, though a good approximation in the lateral direction (Pasquill and Smith, 1983), a simple Gaussian profile is not found in the vertical direction for a ground-level release or for an elevated release under unstable conditions (Deardorff and Willis, 1975; Willis and Deardorff, 1976; Nieuwstadt and van Ulden, 1978; Gryning *et al.*, 1983; Briggs, 1985). Hinrichsen (1986) compared a non-Gaussian analytical model which uses power-law profiles to represent vertical variations in wind speed and turbulence with three Gaussian-type models, and found that the non-Gaussian model agreed better with the observed data.

Analytical solutions of the atmospheric diffusion equation with wind speed and eddy diffusivities expressed as power functions of height have been sporadically found in the literature since the 1950s. Early solutions (Rounds, 1955; Smith, 1957; Walters, 1957) either dealt with only two dimensions, used restricted conjugate power laws, or neglected the inversion effect. By utilizing the Green's function method (Stakgold, 1968), Yeh (1975) derived solutions for the three-dimensional advection–diffusion equation under boundary conditions corresponding to both bounded (inversion) and unbounded (infinite mixing layer) domains. These equations have subsequently been used in dispersion models for an unbounded domain (Yeh and Huang, 1975; Huang, 1979). Demuth's work (1978), which corrected a mistake in Rounds' solutions (1955) for a bounded domain, also has been utilized in dispersion modeling (Tirabassi *et al.*, 1986; Tirabassi, 1989). All of these applications have been restricted to a single isolated point source located at the origin, and limited to total reflection at the boundaries (Neumann-type boundary conditions).

Since actual ground-level concentrations of air pollutants most often fall between Dirichlet (total adsorption) and Neumann types, and since multiple-source dispersion modeling is still done almost exclusively using the Gaussian plume model, the objectives of this paper are two-fold: (1) to systematically derive the solutions of the atmospheric diffusion equation for several boundary condition types; and (2) to apply the Green's function concept (Roach, 1970; Greenberg, 1971; Stakgold, 1979; Beck *et al.*, 1992) to the multiple-source problem, where the sources can be located anywhere in the region of interest.

2. ATMOSPHERIC DIFFUSION EQUATION

The steady-state transport of a non-reactive contaminant released continuously from n point sources located at $(x_S^1, y_S^1, z_S^1), (x_S^2, y_S^2, z_S^2), \ldots, (x_S^n, y_S^n, z_S^n)$ in a Cartesian coordinate system can be described by the following partial differential equation:

$$U(z)\frac{\partial C(x,y,z)}{\partial x} = \frac{\partial}{\partial y}\left(K_y(x,z)\frac{\partial C(x,y,z)}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z(z)\frac{\partial C(x,y,z)}{\partial z}\right) + \sum_{i=1}^{n} Q^i\delta(x-x_S^i)\,\delta(y-y_S^i)\delta(z-z_S^i)$$

where $C(x, y, z)$ is the ambient concentration of the contaminant, Q^i is the emission strength of the ith source located at (x_S^i, y_S^i, z_S^i), and δ is the Dirac delta function. In deriving the above *atmospheric diffusion equation*, the wind is assumed to be blowing in the x-direction, the turbulent fluxes are approximated by gradient transport (K-theory), and the turbulent diffusion in the wind direction is neglected compared to advection (slender plume approximation). These approximations are valid when the scale of the turbulent transport is smaller than the plume dimensions (Pasquill and Smith, 1983; Seinfeld, 1986). In addition, wind speed, $U(z)$, and the eddy diffusivities, $K_y(x, z)$ and $K_z(z)$, are assumed to vary with height, and will be approximated by the following power laws:

$$U(z) = U(z_r)\left(\frac{z}{z_r}\right)^{\alpha} = az^{\alpha}, \quad a = \frac{U(z_r)}{z_r^{\alpha}}$$

$$K_z(z) = K_z(z_r)\left(\frac{z}{z_r}\right)^{\beta} = bz^{\beta}, \quad b = \frac{K_z(z_r)}{z_r^{\beta}}$$

$$K_y(x, z) = f(x)z^{\gamma}$$

where $U(z_r)$ and $K_z(z_r)$ are the measured wind speed and vertical eddy diffusivity at a reference height z_r, $f(x)$ is any integrable function of x, and a, b, α, β, γ are constants that depend on atmospheric stability and surface roughness.

3. BOUNDARY CONDITIONS

Depending on the interactions between the plume emissions and the bounding surfaces, the homogeneous boundary conditions can be of four types (where H is the height of inversion layer):

Neumann type (total reflection)

$$K_z(z)\frac{\partial C(x, y, z)}{\partial z} = 0 \quad \text{at } z = 0$$

$$K_z(z)\frac{\partial C(x, y, z)}{\partial z} = 0 \quad \text{at } z = H$$

Dirichlet type (total adsorption)

$$C(x, y, z) = 0 \quad \text{at } z = 0$$

$$C(x, y, z) = 0 \quad \text{at } z = H$$

Mixed type I

$$K_z(z)\frac{\partial C(x, y, z)}{\partial z} = 0 \quad \text{at } z = 0$$

$$C(x, y, z) = 0 \quad \text{at } z = H$$

Mixed type II

$$C(x, y, z) = 0 \quad \text{at } z = 0$$

$$K_z(z)\frac{\partial C(x, y, z)}{\partial z} = 0 \quad \text{at } z = H$$

Representative vertical concentration profiles corresponding to these boundary types are shown in Fig. 1. The Neumann boundary type states that the earth's surface and inversion layer are both impermeable. When a dispersing plume makes contact with these boundaries, it reflects back into the atmosphere (total reflection). In other words, no adsorption takes place at the boundaries and all the contaminants must therefore exist somewhere between 0 and H. The Dirichlet boundary type, on the other hand, indicates that contaminants are removed immediately upon contact with the boundaries (Monin, 1959), resulting in a significant concentration gradient in the vertical direction (infinite adsorption). The mixed boundary type II simulates a perfectly adsorbing ground and a perfectly reflecting inversion layer, while the mixed boundary type I (included for completeness) corresponds to the reverse situation. It should be noted that in reality, due to partial adsorption, the boundary conditions (both at ground-level and the inversion layer) should be of the Robin type, i.e., a combination of partial reflection and partial adsorption (Calder, 1961; Smith, 1962; Scriven and Fisher, 1975; Horst, 1977; Horst and Slinn, 1984; Chrysikopoulos *et al.*, 1992b); hence, the actual concentrations near the boundaries should fall inbetween the Neumann and Dirichlet cases. We will not evaluate the Robin boundary type in this paper, but will consider it in a separate paper.

The other boundary conditions stipulate that concentrations drop to zero far away from the domain:

$$C(\infty, y, z) = 0, \quad C(x, \pm\infty, z) = 0, \quad C(x, y, \infty) = 0.$$

4. ANALYTICAL SOLUTIONS

The above boundary value problem for a nonreactive pollutant can be solved analytically if $\alpha = \gamma$ (i.e., U and K_y have the same, but arbitrary, power-law dependence on height). Following the procedures of Yeh and Huang (1975) and Yeh (1975) for a single isolated source, the solutions are comprised of three components: a source strength Q^i, a vertical dispersion factor $G_z^i(x, z; x_S^i, z_S^i)$, and a crosswind dispersion factor $G_y^i(x, y; x_S^i, y_S^i)$:

$$C(x, y, z) = \sum_{i=1}^{n} Q^i G_z^i(x, z; x_S^i, z_S^i) G_y^i(x, y; x_S^i, y_S^i) \qquad (1)$$

where the two dispersion factors, G_z^i and G_y^i, are both equal to zero if $x < x_S^i$ (see the Appendix). The multiplication of these two factors (or sub-Green functions) constitutes a Green's function (Greenberg, 1971; Stakgold, 1979), which can be viewed as the concentration response at (x, y, z) due to a unit disturbance (source) at (x_S^i, y_S^i, z_S^i) (Deng and Horne, 1993). The total concentration is therefore the sum of all the responses from the n various sources. The sub-Green functions corresponding to the four types of boundary conditions are systematically presented in the following sections. For convenience, each of the sub-Green function equations will be designated by one or two capital letters: ZR, ZA, and ZM will refer to vertical sub-Green functions that assume total reflection, total adsorption, and mixed boundary conditions, respectively; and Y will refer to crosswind sub-Green functions.

4.1. *Vertical sub-Green functions $G_z^i(x, z; x_S^i, z_S^i)$ of the Neumann type (total reflection)*

4.1.1. *Bounded region (inversion layer).* If an inversion layer is present at height H and adsorption is

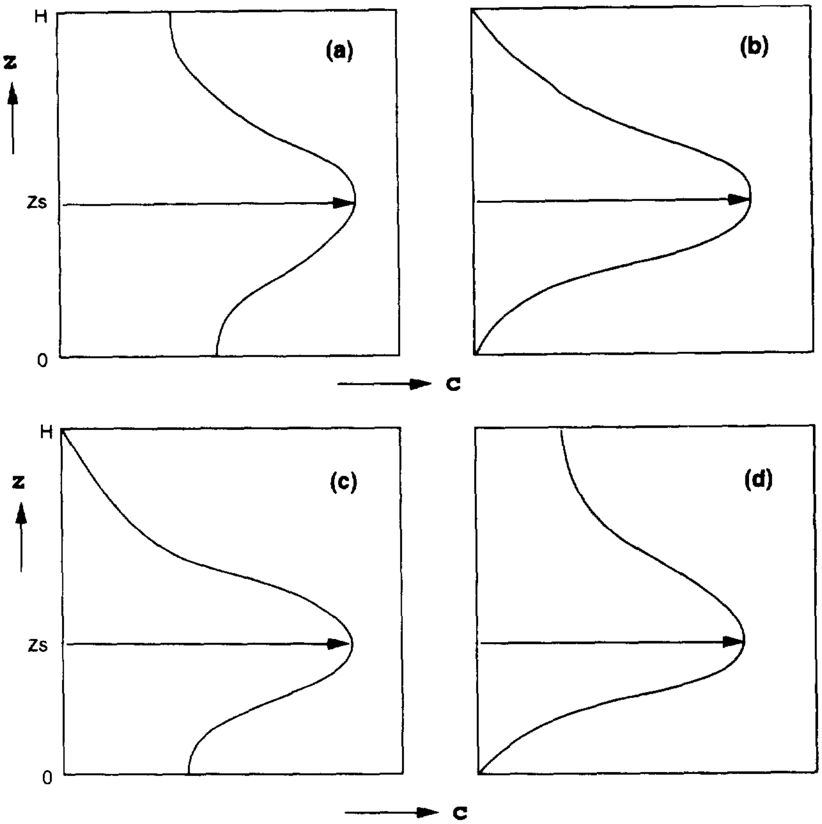

Fig. 1. Representative concentration profiles in vertical direction for different boundary types: (a) Neumann-type (total reflection); (b) Dirichlet-type (total adsorption); (c) mixed-type I; (d) mixed type II.

negligible at both the ground and the inversion layer, the sub-Green function (derived in the Appendix) is

$$G_z^i(x, z; x_S^i, z_S^i) = \frac{\alpha+1}{aH^{\alpha+1}} + \frac{\alpha-\beta+2}{aH^{\alpha-\beta+2}}(zz_S^i)^{(1-\beta)/2}$$

$$\times \sum_{j=1}^{\infty} \frac{J_{-\mu}\left[\lambda_j\left(\frac{z}{H}\right)^{(\alpha-\beta+2)/2}\right] J_{-\mu}\left[\lambda_j\left(\frac{z_S^i}{H}\right)^{(\alpha-\beta+2)/2}\right]}{J_{-\mu}^2(\lambda_j)}$$

$$\times \exp\left[-\frac{b(\alpha-\beta+2)^2\lambda_j^2(x-x_S^i)}{4aH^{\alpha-\beta+2}}\right] \qquad (\mathrm{ZR}_1)$$

with

$$\mu = \frac{1-\beta}{\alpha-\beta+2} \qquad (2)$$

where $J_{-\mu}$ is the Bessel function of the first kind of order $-\mu$; and λ_j, the eigenvalues generated by the homogeneous boundary condition at H, are the zeros (or roots) of the following equation:

$$J_{-\mu+1}(\lambda_j) = 0. \qquad (3)$$

For a ground-level source, which is often of environmental interest, setting $z_S^i = 0$ in equation (ZR_1), and using the limiting form (equation (4)) for small arguments of the Bessel function (Abramowitz and Stegun, 1970) leads to equation (ZR_2):

$$J_\nu(\eta) \to \frac{\eta^\nu}{2^\nu\Gamma(1+\nu)} \quad \text{as } \eta \to 0 \qquad (4)$$

$$G_z^i(x, z; x_S^i, z_S^i = 0) = \frac{\alpha+1}{aH^{\alpha+1}}$$

$$+ \frac{2^\mu(\alpha-\beta+2)}{a\Gamma\left(\frac{\alpha+1}{\alpha-\beta+2}\right)H^{(2\alpha-\beta+3)/2}}(z^{(1-\beta)/2})$$

$$\times \sum_{j=1}^{\infty} \frac{J_{-\mu}\left[\lambda_j\left(\frac{z}{H}\right)^{(\alpha-\beta+2)/2}\right]}{\lambda_j^\mu J_{-\mu}^2(\lambda_j)}$$

$$\times \exp\left[-\frac{b(\alpha-\beta+2)^2\lambda_j^2(x-x_S^i)}{4aH^{\alpha-\beta+2}}\right] \qquad (\mathrm{ZR}_2)$$

where Γ is the gamma function, defined by the following integral:

$$\Gamma(\eta) = \int_0^\infty \tau^{\eta-1}e^{-\tau}\,\mathrm{d}\tau \qquad (5)$$

4.1.2. *Unbounded region (no inversion layer)*. If there is no inversion layer (i.e., $H \to \infty$), the summation over j in equation (ZR_1) becomes an integral (Robson, 1983). By using the following identity (Smith, 1957; Özisik, 1968):

$$\int_0^\infty e^{-\tau^2 k_3} J_\mu(k_1\tau) J_\mu(k_2\tau)\tau \, d\tau = \frac{1}{2k_3}\exp\left(-\frac{k_1^2+k_2^2}{4k_3}\right) \times I_\mu\left(\frac{k_1k_2}{2k_3}\right) \tag{6}$$

equation (ZR_1) reduces to equation (ZR_3):

$$G_z^i(x,z;x_S^i,z_S^i) = \frac{(zz_S^i)^{(1-\beta)/2}}{b(\alpha-\beta+2)(x-x_S^i)} \times I_{-\mu}\left[\frac{2a(zz_S^i)^{(\alpha-\beta+2)/2}}{b(\alpha-\beta+2)^2(x-x_S^i)}\right] \times \exp\left[-\frac{a(z^{\alpha-\beta+2}+z_S^{i\,\alpha-\beta+2})}{b(\alpha-\beta+2)^2(x-x_S^i)}\right] \tag{ZR$_3$}$$

where $I_{-\mu}$ is the modified Bessel function of the first kind of order $-\mu$. Similar to the bounded region, for a ground-level source in an unbounded atmosphere, setting $z_S^i = 0$ in equation (ZR_3), and using the limiting form (equation (7)) for small arguments of the modified Bessel function (Abramowitz and Stegun, 1970) yields equation (ZR_4):

$$I_\nu(\eta) \to \frac{\eta^\nu}{2^\nu \Gamma(1+\nu)} \quad \text{as} \quad \eta \to 0 \tag{7}$$

$$G_z^i(x,z;x_S^i,z_S^i=0) = \frac{\alpha-\beta+2}{a\Gamma\left(\dfrac{\alpha+1}{\alpha-\beta+2}\right)} \times \left[\frac{a}{b(\alpha-\beta+2)^2(x-x_S^i)}\right]^{(\alpha+1)/(\alpha-\beta+2)} \times \exp\left[-\frac{az^{\alpha-\beta+2}}{b(\alpha-\beta+2)^2(x-x_S^i)}\right]. \tag{ZR$_4$}$$

4.2. *Vertical sub-Green functions $G_z^i(x,z;x_S^i,z_S^i)$ of the Dirichlet type (total adsorption)*

4.2.1. *Bounded region (inversion layer)*. For a case where total adsorption occurs at both boundaries, the vertical sub-Green function (derived in the Appendix) becomes:

$$G_z^i(x,z;x_S^i,z_S^i) = \frac{\alpha-\beta-2}{aH^{\alpha-\beta+2}}(zz_S^i)^{(1-\beta)/2} \times \sum_{j=1}^\infty \frac{J_\mu\left[\lambda_j\left(\dfrac{z}{H}\right)^{(\alpha-\beta+2)/2}\right] J_\mu\left[\lambda_j\left(\dfrac{z_S^i}{H}\right)^{(\alpha-\beta+2)/2}\right]}{J_{\mu+1}^2(\lambda_j)} \times \exp\left[-\frac{b(\alpha-\beta+2)^2\lambda_j^2(x-x_S^i)}{4aH^{\alpha-\beta+2}}\right] \tag{ZA$_1$}$$

where $J_{\mu+1}^2(\lambda_j)$ can be replaced by either $J_{\mu-1}^2(\lambda_j)$ or $J_\mu'^2(\lambda_j)$ (see Appendix). The eigenvalues of the system, λ_j, are the roots of the following equation:

$$J_\mu(\lambda_j) = 0. \tag{8}$$

The differences between the sub-Green function for this case and that of the Neumann case (equation (ZR_1)) are the order of the Bessel function (μ vs $-\mu$), the square term in the denominator ($\mu + 1$ vs $-\mu$), no first term before the summation, and the eigenvalue equation.

4.2.2. *Unbounded region (no inversion layer)*. Letting $H \to \infty$ in equation (ZA_1) and using identity (6) lead to the sub-Green function for the Dirichlet unbounded region:

$$G_z^i(x,z;x_S^i,z_S^i) = \frac{(zz_S^i)^{(1-\beta)/2}}{b(\alpha-\beta+2)(x-x_S^i)} \times I_\mu\left[\frac{2a(zz_S^i)^{(\alpha-\beta+2)/2}}{b(\alpha-\beta+2)^2(x-x_S^i)}\right] \times \exp\left[-\frac{a(z^{\alpha-\beta+2}+z_S^{i\,\alpha-\beta+2})}{b(\alpha-\beta+2)^2(x-x_S^i)}\right]. \tag{ZA$_2$}$$

Equation (ZA_2) differs from equation (ZR_3) only in the order of the modified Bessel function (μ vs $-\mu$). The sub-Green function for a ground-level release under a Dirichlet-type boundary condition cannot be found using this type of approach. Yih (1952) and Panchev (1985) presented solutions for a similar (surface release) problem with a constant (but non-zero) Dirichlet boundary condition at the ground. Also, it is important that equation (ZA_2) not be used for the constant turbulent flux layer (i.e. $\beta = 1$ or $\mu = 0$) commonly found in neutral stability atmosphere, because an incorrect boundary behavior will be obtained, due to the fact that at the origin, the modified Bessel function of the first kind of zero order is different from its other orders (i.e., $I_0(0) = 1$, whereas $I_\mu(0) = 0$ for $\mu \neq 0$).

4.3. *Vertical sub-Green functions $G_z^i(x,z;x_S^i,z_S^i)$ of the mixed type I and II*

The sub-Green functions for a bounded region with the mixed type I and the mixed type II boundary conditions have also been derived and are listed, along with the Neumann and Dirichlet boundary types, in Table 1 for brevity and comparison. By inspection, it can be confirmed that the four sub-Green functions in Table 1 exhibit the correct asymptotic behavior (i.e., zero flux or zero concentration) at the boundaries. In addition, if the domain is unbounded, equations (ZM_1) and (ZM_2) reduce to equations (ZR_3) and (ZA_2), respectively, while for a ground-level release, equation (ZM_1) reduces to equation (ZR_4) as described previously.

Table 1. Vertical sub-Green functions within a bounded region for different boundary types

Type	Conditions	Sub-Green function and eigenvalues
Neumann	$K_z(z)\dfrac{\partial C(x,y,z)}{\partial z}=0$ at $z=0$ $K_z(z)\dfrac{\partial C(x,y,z)}{\partial z}=0$ at $z=H$	Equation (ZR_1): $G_z^i(x,z;x_S^i,z_S^i)=\dfrac{\alpha+1}{aH^{\alpha+1}}+\dfrac{\alpha-\beta+2}{aH^{\alpha-\beta+2}}(zz_S^i)^{(1-\beta)/2}\times\sum_{j=1}^{\infty}\dfrac{J_{-\mu}[\lambda_j(\frac{z}{H})^{(\alpha-\beta+2)/2}]J_{-\mu}[\lambda_j(\frac{z_S^i}{H})^{(\alpha-\beta+2)/2}]}{J_{-\mu}^2(\lambda_j)}\times\exp\left[-\dfrac{b(\alpha-\beta+2)^2\lambda_j^2(x-x_S^i)}{4aH^{\alpha-\beta+2}}\right]$ $J_{-\mu+1}(\lambda_j)=0$
Dirichlet	$C(x,y,z)=0$ at $z=0$ $C(x,y,z)=0$ at $z=H$	Equation (ZA_1): $G_z^i(x,z;x_S^i,z_S^i)=\dfrac{\alpha-\beta+2}{aH^{\alpha-\beta+2}}(zz_S^i)^{(1-\beta)/2}\times\sum_{j=1}^{\infty}\dfrac{J_{\mu}[\lambda_j(\frac{z}{H})^{(\alpha-\beta+2)/2}]J_{\mu}[\lambda_j(\frac{z_S^i}{H})^{(\alpha-\beta+2)/2}]}{J_{\mu+1}^2(\lambda_j)}\times\exp\left[-\dfrac{b(\alpha-\beta+2)^2\lambda_j^2(x-x_S^i)}{4aH^{\alpha-\beta+2}}\right]$ $J_{\mu}(\lambda_j)=0$
Mixed I	$K_z(z)\dfrac{\partial C(x,y,z)}{\partial z}=0$ at $z=0$ $C(x,y,z)=0$ at $z=H$	Equation (ZM_1): $G_z^i(x,z;x_S^i,z_S^i)=\dfrac{\alpha-\beta+2}{aH^{\alpha-\beta+2}}(zz_S^i)^{(1-\beta)/2}\times\sum_{j=1}^{\infty}\dfrac{J_{-\mu}[\lambda_j(\frac{z}{H})^{(\alpha-\beta+2)/2}]J_{-\mu}[\lambda_j(\frac{z_S^i}{H})^{(\alpha-\beta+2)/2}]}{J_{-\mu+1}^2(\lambda_j)}\times\exp\left[-\dfrac{b(\alpha-\beta+2)^2\lambda_j^2(x-x_S^i)}{4aH^{\alpha-\beta+2}}\right]$ $J_{-\mu}(\lambda_j)=0$
Mixed II	$C(x,y,z)=0$ at $z=0$ $K_z(z)\dfrac{\partial C(x,y,z)}{\partial z}=0$ at $z=H$	Equation (ZM_2): $G_z^i(x,z;x_S^i,z_S^i)=\dfrac{\alpha-\beta+2}{aH^{\alpha-\beta+2}}(zz_S^i)^{(1-\beta)/2}\times\sum_{j=1}^{\infty}\dfrac{J_{\mu}[\lambda_j(\frac{z}{H})^{(\alpha-\beta+2)/2}]J_{\mu}[\lambda_j(\frac{z_S^i}{H})^{(\alpha-\beta+2)/2}]}{J_{\mu}^2(\lambda_j)}\times\exp\left[-\dfrac{b(\alpha-\beta+2)^2\lambda_j^2(x-x_S^i)}{4aH^{\alpha-\beta+2}}\right]$ $J_{\mu-1}(\lambda_j)=0$

4.4. *Crosswind sub-Green functions* $G_y^i(x,y;x_S^i,y_S^i)$

The crosswind sub-Green function takes the following form (Yeh, 1975; details in the Appendix):

$$G_y^i(x,y;x_S^i,y_S^i)=\frac{\sqrt{a}}{\sqrt{4\pi(X-X_S^i)}}\times\exp\left[-\frac{a(y-y_S^i)^2}{4(X-X_S^i)}\right] \tag{9}$$

with

$$X=\int_{x_c}^{x}f(\tau)\,d\tau\,,\qquad X_S^i=\int_{x_c}^{x_S^i}f(\tau)\,d\tau$$

where x_c is some reference location in Cartesian coordinates. Since the power in the exponential term is 2, the crosswind concentrations are symmetrical with respect to y_S^i.

4.4.1. *Point source* ($Q^i=Q_P^i$). All the sub-Green functions presented above are exact solutions to the steady-state atmospheric diffusion equation, assuming that the lateral eddy diffusivity and wind speed have the same but arbitrary power-law variation with height, and that the lateral eddy diffusion also depends on the downwind distance via $f(x)$. To find an explicit expression for $f(x)$, the two parameters can be further related to each other as follows (as is done frequently in air dispersion modeling):

$$K_y = \tfrac{1}{2} U \frac{d\sigma_y^2}{dx}. \tag{10}$$

Substituting equation (10) into the power-law equation for K_y gives $f(x) = \frac{1}{2} a\, d\sigma_y^2/dx$. For multiple source modeling, we will assume that $f(x)$ depends on the *relative* distance between the source and the location of interest (i.e., $x - x_S^i$), but not on the downwind distance alone. The integration limits of $X - X_S^i$ in equation (9) then become from 0 to $x - x_S^i$ (instead of from x to x_S^i). Carrying out the integration, a point source sub-Green function is obtained:

$$G_y^i(x, y; x_S^i, y_S^i) = \frac{1}{\sqrt{2\pi}\sigma_y(x - x_S^i)} \times \exp\left[-\frac{(y - y_S^i)^2}{2\sigma_y^2(x - x_S^i)}\right] \tag{Y$_1$}$$

where $\sigma_y(x - x_S^i)$ and $\sigma_y^2(x - x_S^i)$ are the standard deviation (diffusion coefficient) and the mean square particle displacement evaluated at $x - x_S^i$, respectively.

4.4.2. *Finite line source* ($Q^i = Q_\ell^i$). A line source can be considered as a superposition of point sources, each having an emission strength $Q_\ell\, d|\mathbf{q}|$ (Csanady, 1972), where Q_ℓ is the unit strength and $d|\mathbf{q}|$ is a differential source length. Thus, for a continuous finite line source extending from $(x_{S1_1}, y_{S1_1}, z_{S1_1})$ to $(x_{S1_2}, y_{S1_2}, z_{S1_2})$ in Cartesian coordinates, the summation in equation (1) becomes an integration over the length of the line source, B. For example, for the point source kernel (Y$_1$):

$$C(x, y, z) = Q_\ell \int_0^B G_z(x, z; x_S(\mathbf{q}), z_S(\mathbf{q})) \times \frac{1}{\sqrt{2\pi}\sigma_y(x - (x_S(\mathbf{q}))} \times \exp\left[-\frac{(y - y_S(\mathbf{q}))^2}{2\sigma_y^2(x - x_S(\mathbf{q}))}\right] d|\mathbf{q}| \tag{11}$$

where

$$B = |\mathbf{q}| = \sqrt{(x_{S1_1} - x_{S1_2})^2 + (y_{S1_1} - y_{S1_2})^2 + (z_{S1_1} - z_{S1_2})^2} \tag{12}$$

The above integral can be carried out explicitly for a few simplified cases. If the line source is perpendicular to the wind direction ($x_{S1_1} = x_{S1_2} = x_S$), and has a constant elevation ($z_{S1_1} = z_{S1_2} = z_S$), G_z can be pulled out of the integral and the result of the integration yields a sub-Green function (equation (Y$_2$)). Note that index i is again included below to indicate the ith line source, and erf represents the error function, defined by (13):

$$G_y^i(x, y; x_S^i, y_S^i) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{y_{S1_2}^i - y}{\sqrt{2}\,\sigma_y(x - x_S^i)}\right) - \operatorname{erf}\left(\frac{y_{S1_1}^i - y}{\sqrt{2}\,\sigma_y(x - x_S^i)}\right)\right] \tag{Y$_2$}$$

$$\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\tau^2} d\tau. \tag{13}$$

Furthermore, if the x-axis passes through the mid-points of B^i (i.e., $y_{S1_2}^i = B^i/2$ and $y_{S1_1}^i = -B^i/2$), equation (Y$_2$) becomes:

$$G_y^i(x, y; x_S^i, y_S^i) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{B^i/2 - y}{\sqrt{2}\,\sigma_y(x - x_S^i)}\right) + \operatorname{erf}\left(\frac{B^i/2 + y}{\sqrt{2}\sigma_y(x - x_S^i)}\right)\right]. \tag{Y$_3$}$$

4.4.3. *Infinite line source* ($Q^i = Q_\ell^i$). For continuous infinite line sources, $y_{S1_2}^i = \infty$, $y_{S1_1}^i = -\infty$, and equation (Y$_2$) reduces to

$$G_y^i(x, y; x_S^i, y_S^i) = 1. \tag{Y$_4$}$$

This is equivalent to solving the two-dimensional case:

$$U(z)\frac{\partial C(x, z)}{\partial x} = \frac{\partial}{\partial z}\left(K_z(z)\frac{\partial C(x, z)}{\partial z}\right) + \sum_{i=1}^{n} \delta(x - x_S^i)\delta(z - z_S^i).$$

4.4.4. *Finite area source* ($Q^i = Q_a^i$). For an area source extending from y_{S1_1} to y_{S1_2} in the crosswind direction and from x_{S1_1} to x_{S1_2} in the downwind direction, the concentration at (x, y, z) is calculated via the superposition of line sources:

$$C(x, y, z) = Q_a G_z(x, z; x_S^i, z_S^i) \int_{x_{S1_1}}^{x_{S1_2}} G_y(Y_2) d|\mathbf{q}| \tag{14}$$

where Q_a is the emission strength per unit area, $d|\mathbf{q}|$ is the differential downwind source length, $G_y^i(Y_2)$ is equation (Y$_2$), and $G_z(x, z; x_S^i, z_S^i)$ is any vertical sub-Green function described in the previous sections. The integral in equation (14) is a sub-Green function for an area release:

$$G_y^i(x, y; x_S^i, y_S^i) = \int_{x_{S1_1}}^{x_{S1_2}} G_y(Y_2) d|\mathbf{q}|. \tag{Y$_5$}$$

4.5. *Gaussian plume models*

The Gaussian plume equation is a special case where wind speed and eddy diffusivities are assumed constant with height (i.e., $\alpha = \beta = 0$, $\mu = \frac{1}{2}$). For example, consider total reflection or total adsorption in a bounded region where an inversion layer is at H.

The following identities for a Bessel function of order $\pm\frac{1}{2}$ can be used:

$$J_{\frac{1}{2}}(\eta) = \sqrt{\frac{2}{\pi\eta}}\sin\eta \tag{15}$$

$$J_{-\frac{1}{2}}(\eta) = \sqrt{\frac{2}{\pi\eta}}\cos\eta\,. \tag{16}$$

The eigenvalue equations (3) and (8) both give $\lambda_j = j\pi$, and equation (ZR_1) and equation (ZA_1) reduce to equation (ZG_1) (as appears in Seinfeld, 1986) and equation (ZG_2), respectively (where ZG denotes a vertical sub-Green function of the Gaussian type):

$$G_z^i(x, z; x_S^i, z_S^i) = \frac{1}{aH} + \frac{2}{aH}\sum_{j=1}^{\infty}\cos\left(\frac{j\pi z}{H}\right) \times\cos\left(\frac{j\pi z_S^i}{H}\right)\exp\left[-\frac{bj^2\pi^2(x-x_S^i)}{aH^2}\right]. \tag{ZG_1}$$

$$G_z^i(x, z; x_S^i, z_S^i) = \frac{2}{aH}\sum_{j=1}^{\infty}\sin\left(\frac{j\pi z}{H}\right) \times\sin\left(\frac{j\pi z_S^i}{H}\right)\exp\left[-\frac{bj^2\pi^2(x-x_S^i)}{aH^2}\right]. \tag{ZG_2}$$

Likewise, for the mixed boundary types, equations (ZM_1) and (ZM_2) reduce to equations (ZG_3) and (ZG_4) (listed in Table 2), respectively. For an unbounded region, identities for a modified Bessel function of order $\pm\frac{1}{2}$ can be used:

$$I_{-\frac{1}{2}}(\eta) = \sqrt{\frac{2}{\pi\eta}}\cosh\eta = \sqrt{\frac{2}{\pi\eta}}\left(\frac{e^{\eta}+e^{-\eta}}{2}\right) \tag{17}$$

$$I_{\frac{1}{2}}(\eta) = \sqrt{\frac{2}{\pi\eta}}\sinh\eta = \sqrt{\frac{2}{\pi\eta}}\left(\frac{e^{\eta}-e^{-\eta}}{2}\right) \tag{18}$$

Using equations (17) and (18), equation (ZR_3) and equation (ZA_2) reduce to

$$G_z^i(x, z; x_S^i, z_S^i) = \frac{1}{\sqrt{4\pi ab(x-x_S^i)}}\left\{\exp\left[-\frac{a(z-z_S^i)^2}{4b(x-x_S^i)}\right] \pm\exp\left[-\frac{a(z+z_S^i)^2}{4b(x-x_S^i)}\right]\right\} \tag{ZG_5}$$

where the positive sign corresponds to the total reflection case while the negative sign corresponds to the total adsorption case. Combining equation (ZG_5), equation (Y_1) and point source strength Q_p yields the familiar Gaussian plume equation.

Table 2. Vertical sub-Green functions within a bounded region for Gaussian plume types[a]

Type	Conditions	Sub-Green function
Neumann	$K_z\frac{\partial C(x,y,z)}{\partial z}=0$ at $z=0$; $K_z\frac{\partial C(x,y,z)}{\partial z}=0$ at $z=H$	Equation (ZG_1): $G_z^i(x,z;x_S^i,z_S^i)=\frac{1}{aH}+\frac{2}{aH}\times\sum_{j=1}^{\infty}\cos\left(\frac{j\pi z}{H}\right)\cos\left(\frac{j\pi z_S^i}{H}\right)\exp\left[-\frac{bj^2\pi^2(x-x_S^i)}{aH^2}\right]$
Dirichlet	$C(x,y,z)=0$ at $z=0$; $C(x,y,z)=0$ at $z=H$	Equation (ZG_2): $G_z^i(x,z;x_S^i,z_S^i)=\frac{2}{aH}\times\sum_{j=0}^{\infty}\sin\left(\frac{j\pi z}{H}\right)\sin\left(\frac{j\pi z_S^i}{H}\right)\exp\left[-\frac{bj^2\pi^2(x-x_S^i)}{aH^2}\right]$
Mixed I	$K_z\frac{\partial C(x,y,z)}{\partial z}=0$ at $z=0$; $C(x,y,z)=0$ at $z=H$	Equation (ZG_3): $G_z^i(x,z;x_S^i,z_S^i)=\frac{2}{aH}\sum_{j=0}^{\infty}\cos\left(\frac{(j+\frac{1}{2})\pi z}{H}\right)\cos\left(\frac{(j+\frac{1}{2})\pi z_S^i}{H}\right)\times\exp\left[-\frac{b(j+\frac{1}{2})^2\pi^2(x-x_S^i)}{aH^2}\right]$
Mixed II	$C(x,y,z)=0$ at $z=0$; $K_z\frac{\partial C(x,y,z)}{\partial z}=0$ at $z=H$	Equation (ZG_4): $G_z^i(x,z;x_S^i,z_S^i)=\frac{2}{aH}\sum_{j=0}^{\infty}\sin\left(\frac{(j+\frac{1}{2})\pi z}{H}\right)\sin\left(\frac{(j+\frac{1}{2})\pi z_S^i}{H}\right)\times\exp\left[-\frac{b(j+\frac{1}{2})^2\pi^2(x-x_S^i)}{aH^2}\right]$

[a] This table (revised from Lin and Hildemann, 1995) is the Gaussian plume counterpart (i.e., $\alpha=\beta=0$, $\mu=\frac{1}{2}$) of Table 1.

The vertical sub-Green functions for the mixed boundary types, equation (ZM_1) and equation (ZM_2) in Table 1, can also be used to derive the Gaussian plume equation using the same approach.

5. DISCUSSION

With the solution of the atmospheric diffusion equation conveniently breaking down into three components—a source term, a vertical sub-Green function, and a crosswind sub-Green function—almost all of the dispersion models currently existing which use height-varying wind speed and eddy diffusivities can be obtained by combining these functions. Table 3 summarizes the existing analytical air dispersion equations which can be represented as special cases of the general solutions given in this paper. For example, the two-dimensional kernel

Table 3. Form and assumptions of existing steady-state analytical air dispersion models, as simplified from the general equation (1), $C(x, y, z) = \sum_{i=1}^{n} Q^i G_z^i(x, z; x_S^i, z_S^i) G_y^i(x, y; x_S^i, y_S^i)$, in present work[a]

Concentration	Assumptions	References
$Q_\ell \times ZR_4 \times Y_4$	(1) Two dimensions (2) Ground-level infinite line source (3) $S = Q_\ell \delta(x)\delta(z)$ (4) No inversion layer (5) Total reflection at $z = 0$	Equation (6), (10), (11) in Yih (1952) Equation (8.11) in Sutton (1953)[b] Equation (2.16) in Walters (1957) Equation (10.117) in Monin and Yaglom (1971) Equation (34) in Liu and Seinfeld (1975) Equation (6) in van Ulden (1978) Equation (6) in Melli and Runca (1979) Equation (3.20) in Pasquill and Smith (1983) Equation (5.17) in Panchev (1985) Equation (11b) in Koch (1989)[b]
$Q_\ell \times ZR_3 \times Y_4$	(1) Two dimensions (2) Elevated infinite line source (3) $S = Q_\ell \delta(x)\delta(z - z_S)$ (4) No inversion layer (5) Total reflection at $z = 0$	Equation (2.17) in Walters (1957) Equation (12) in Huang (1979)
$Q_\ell \times ZR_1 \times Y_4$	(1) Two dimensions (2) Elevated infinite line source (3) $S = Q_\ell \delta(x)\delta(z - z_S)$ (4) Inversion layer at H (5) Total reflection at $z = 0$, $z = H$	Equation (47) in Rounds (1955)[b] Equation (8i) in Yeh and Tsai (1976)[b] Equation (2) in Demuth (1978)[b] Equation (17) in Robson (1983)[b]
$Q_\ell \times ZG_3 \times Y_3$	(1) Gaussian plume (2) Ground-level finite line source (3) $S = Q_\ell \delta(x)\delta(y)\delta(z)$ (4) No inversion layer (5) Total reflection at $z = 0$	Appendix in Csanady (1972) Equation (10.16) in Dobbins (1979)
$Q_p \times ZR_4 \times Y_1$	(1) Three dimensions (2) Ground-level point source (3) $S = Q_p \delta(x)\delta(y)\delta(z)$ (4) No inversion layer (5) Total reflection at $z = 0$	Equation (9) in Huang (1979) Equation (4) in Lehning *et al.* (1994)
$Q_p \times ZR_3 \times Y_1$	(1) Three dimensions (2) Elevated point source (3) $S = Q_p \delta(x)\delta(y)\delta(z - z_S)$ (4) No inversion layer (5) Total reflection at $z = 0$	Berlyand (1975) Equation (6) in Huang (1979) Equation (2) in Hinrichsen (1986)
$Q_p \times ZR_1 \times Y_1$	(1) Three dimensions (2) Elevated point source (3) $S = Q_p \delta(x)\delta(y)\delta(z - z_S)$ (4) Inversion layer at H (5) Total reflection at $z = 0$, $z = H$	Equation (23), (12) in Yeh (1975)[b] Equation (4b), (5) in Tirabassi *et al.* (1986)[b] Tirabassi (1989)[b] p. 21
$Q_a \times ZR_4 \times Y_5$	(1) Three dimensions (2) Ground-level area source (3) Source not necessarily at the origin[c] (4) No inversion layer (5) Partial adsorption at $z = 0$	Equation (9a) in Chrysikopoulos *et al.* (1992a); Equation (29) in Chrysikopoulos *et al.* (1992b)

[a] Except for the Gaussian-type, all the models cited only account for single isolated source (i.e., $n = 1$).
[b] Typographical errors are found in some of the equations in this paper.
[c] For all the other models cited in this table, the source is located at the origin (i.e., $x_S = 0$, $y_S = 0$, with or without $z_S = 0$); for this model, $S = Q_a \int_{x_{S1_1}}^{x_{S1_2}} \int_{-B/2}^{B/2} \delta(x - x_S)\delta(y - y_S)\delta(z - z_S)\,dy_S\,dx_S$.

$(ZR_4)\times(Y_4)$ is the well-known Roberts solution (unpublished paper, cited in Monin and Yaglom, 1971) for a ground-level infinite line source, which was independently derived by Yih (1952) using a similarity method and by Walters (1957) using the Hankel transform. This kernel has been cited extensively in the literature (e.g., Sutton, 1953; Pasquill and Smith, 1983; Panchev, 1985) and has been widely used in air dispersion studies (Liu and Seinfeld, 1975; van Ulden, 1978; Melli and Runca, 1979; Koch, 1989). The three-dimensional kernel $(ZR_3)\times(Y_1)$ is the equation developed by Berlyand (1975) and Huang (1979), and is the analytical model used in Hinrichsen's (1986) comparison study with Gaussian models. Equation $(ZR_4)\times(Y_5)$ is the three-dimensional model utilized by Chrysikopoulos and coworkers (1992a, b), while the model used by Tirabassi and coworkers (1986, 1989) is the combination of (ZR_1) and (Y_1).

As is indicated in Table 3, virtually all of the existing analytical dispersion models inherently assume a perfectly reflecting ground. For contaminants that interact strongly with the earth's surface, such as H_2O_2 absorption into water or SO_2 uptake by vegetation, partial adsorption or deposition is a significant removal process. Hence, these existing models may tend to overestimate airborne concentrations near ground-level. On the other hand, very few of these models include the effect of the inversion layer, so concentrations may be underestimated for elevated releases or for a low inversion height.

6. APPLICATIONS

To illustrate three-dimensional dispersion for multiple sources, an area rarely explored outside of the Gaussian plume equation, consider a simple case where two ground-level point sources with equal strength Q are located at ($x_S^1 = 10$ m, $y_S^1 = 0$ m, $z_S^1 = 0$ m) and ($x_S^2 = 100$ m, $y_S^2 = 0$ m, $z_S^2 = 0$ m) (90 m apart in the downwind direction) in a free and unbounded atmosphere ($H \to \infty$). Assuming the pollutant is totally reflected, equation (1) with kernel $(ZR_4)\times(Y_1)$ and $n = 2$ comprise the solution for this case. For illustrative purposes, meteorological input parameters are taken from Huang (1979) and Chrysikopoulos and coworkers (1992a): $\alpha = 0.29$, $\beta = 0.45$, $a = 1.5$ (m sec^{-1}) (m$^{-0.29}$), $b = 0.025$ (m^2 sec^{-1}) (m$^{-0.45}$), and $\sigma_y(x) = 0.32x^{1/(1+\alpha)}$. Figure 2 shows the normalized vertical concentration profiles due solely to the first source at ($x_S^1 = 10$ m, $y_S^1 = 0$ m, $z_S^1 = 0$ m). At short downwind distances, due to the low wind speed and eddy diffusion coefficients near ground level, plume spreading in the vertical direction is limited. Diminishing concentrations with increasing distance result primarily from horizontal spreading. Because of this, contaminant levels near ground level (i.e. near breathing level) are quite high, making human exposure a major concern near the source. Only beyond a certain distance downwind ($x \geqslant 70$ m) does the plume spread noticeably in the vertical direction (Fig. 2b). The zero gradient behavior at ground level, expected from equation (ZR_4), can be seen at all downwind locations (Fig. 2a).

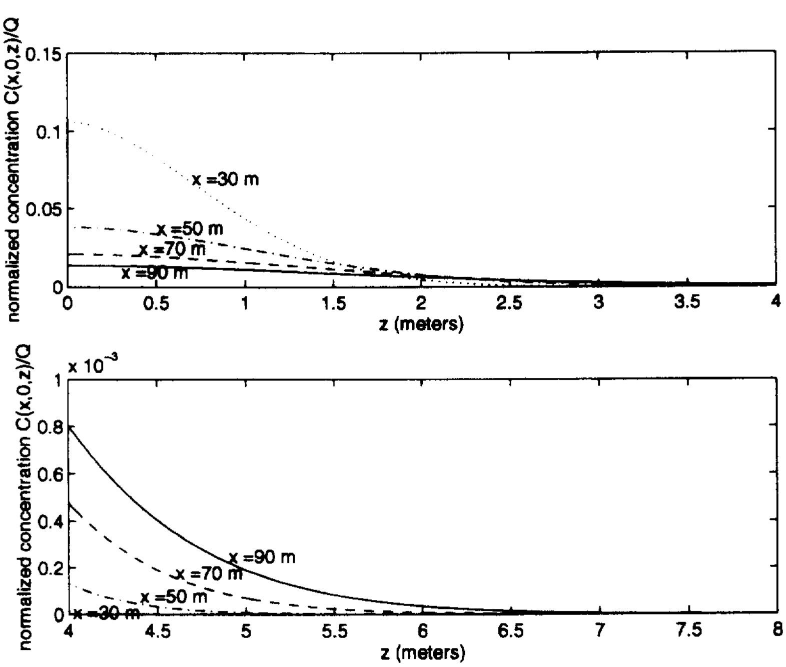

Fig. 2. Variation of normalized centerline concentration, $C(x, 0, z)/Q$, with height due to a single source at (10, 0, 0 m): (a) $0 \leqslant z \leqslant 4$ m; (b) $4 \leqslant z \leqslant 8$ m.

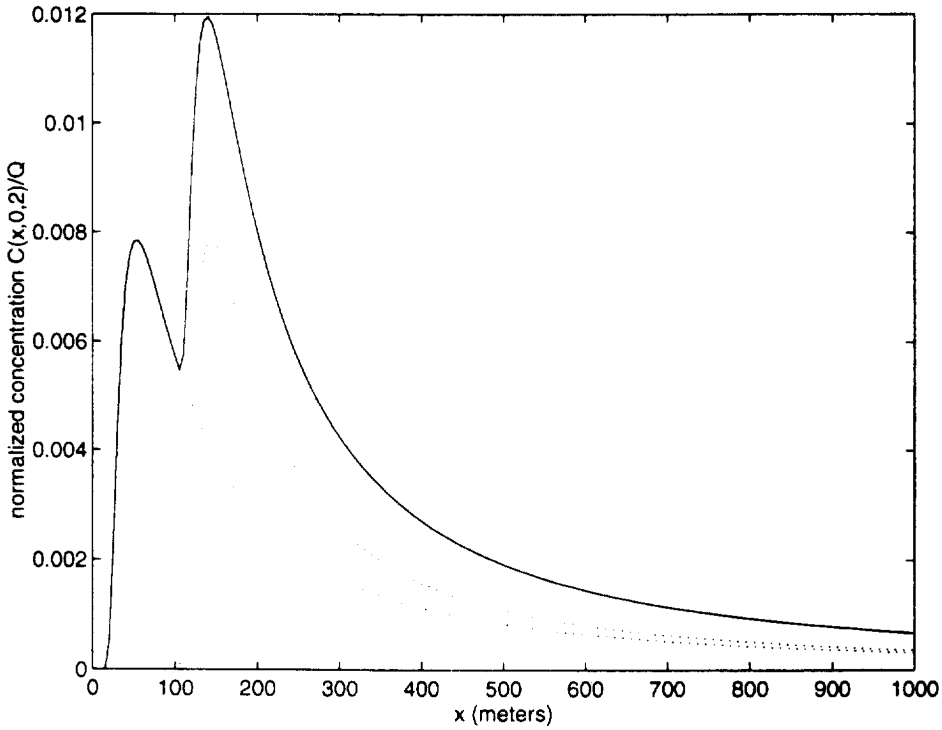

Fig. 3. Variation of normalized breathing level concentration on the centerline, $C(x, 0, 2\ \mathrm{m})/Q$, with downwind distance due to two sources at (10, 0, 0 m) and (100, 0, 0 m).

Figure 3 shows the normalized "breathing level" ($z = 2$ m) concentrations directly downwind of the two sources. The dashed lines represent the individual source contributions, while the solid line shows the sum of the two sources. Between $x = 0$ and 100 m, only the first plume contributes to the concentration: it initially rises and builds up to a maximum level due to turbulent mixing, and then begins decreasing because of continued vertical and horizontal spreading. At $x = 100$ m, the concentration again rises sharply, as the emissions from the first plume join with the second plume.

To examine how plumes disperse in the horizontal plane, consider a second case where two ground-level point sources with equal strength Q are located at ($x_S^1 = 10$ m, $y_S^1 = -20$ m, $z_S^1 = 0$ m) and ($x_S^2 = 10$ m, $y_S^2 = 20$ m, $z_S^2 = 0$ m) (40 m apart in the crosswind direction). Normalized breathing level concentrations in the crosswind direction at two downwind distances, $x = 150$ m and $x = 500$ m, are plotted in Fig. 4 for this case, assuming the pollutant is totally reflected. The resulting contour profiles in the x–y plane ($z = 2$ m plane) are shown in Fig. 5. The two plumes in fact disperse independently, as can be seen from the two distinct Gaussian shapes shown for $x = 150$ m in Fig. 4, and also from the two concentration "eyes" seen in Fig. 5. As the plumes travel further downwind, gradual spreading in the horizontal direction causes the plume concentrations to overlap with each other until a single Gaussian shape is finally formed. An interesting result of the overlap is that, as the downwind distance increases, the locations at which the crosswind maxima occur shift towards the centerline. As a result, the y location that is equidistant between these two sources will experience the highest level of exposure sufficiently far downwind.

7. SUMMARY

Three-dimensional analytical solutions of the atmospheric diffusion equation with height-dependent wind speed and eddy diffusivities can be conveniently broken down into three components: a source strength, a crosswind dispersion factor, and a vertical dispersion factor. The two dispersion factors together constitute a Green's function, which can be viewed as the concentration response due to a unit disturbance (source). The Green's functions corresponding to different homogeneous boundary types are derived in a systematic fashion for point, line and area sources. A methodology based on the superposition of Green's functions is proposed for multiple source dispersion. By choosing appropriate sub-Green functions not only can multiple point, line, and area releases be included, but geographically varying boundary types within the same domain can also be considered (Yeh and Brutsaert, 1970; 1971). The methodology presented is particularly suitable for the evaluation of human exposure to pollutants from multiple sources.

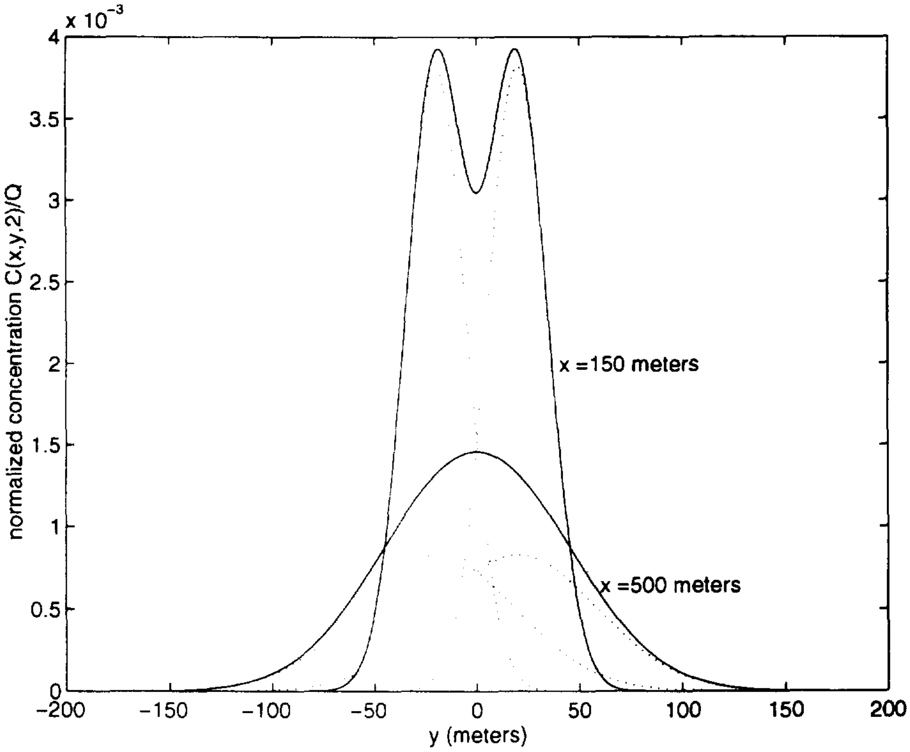

Fig. 4. Variation of normalized breathing level concentration, $C(x, y, 2\ \mathrm{m})/Q$, with crosswind distance due to two side-by-side ground-level sources at (10, -20, 0 m) and (10, 20, 0 m).

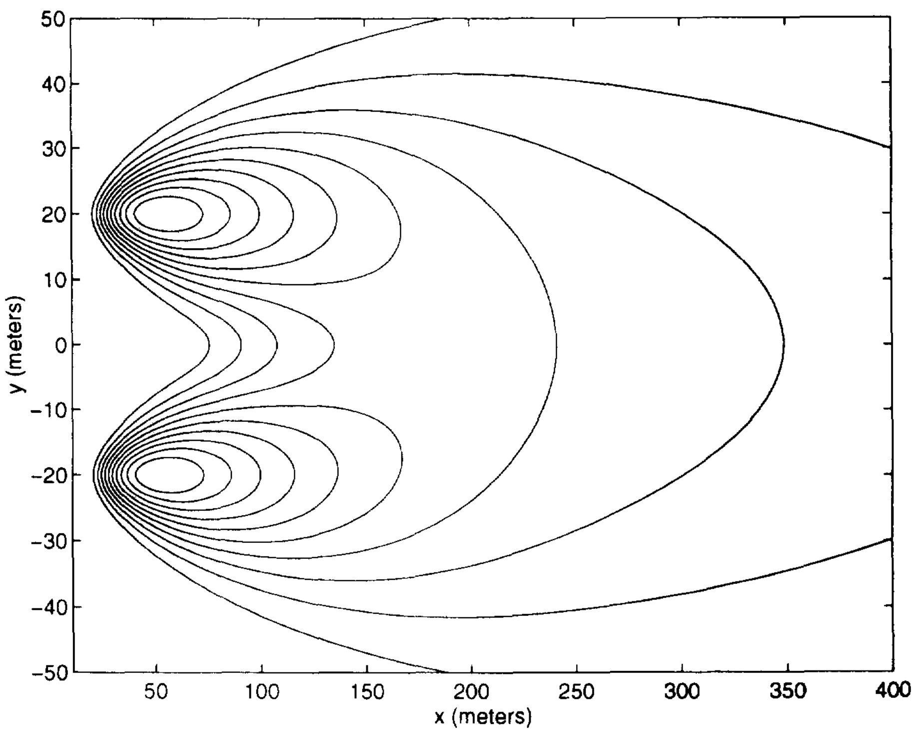

Fig. 5. Normalized breathing level concentration contours, $C(x, y, 2)/Q$, in the x–y plane ($z = 2$ m plane) due to two side-by-side ground-level sources at (10, -20, 0 m) and (10, 20, 0 m).

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APPENDIX

The sub-Green function $G_z(x, z; x_S, z_S)$ within a bounded region for different boundary types will be derived, as well as the sub-Green function $G_y(x, y; x_S, y_S)$. Using the Green's function method (Stakgold, 1968; Greenberg, 1971; Yeh, 1975), the solution of the steady-state atmospheric diffusion equation with homogeneous boundary conditions can be expressed as

$$C(x, y, z) = \int_0^x \int_{-x}^{x} \int_0^H G_z(x_0, z_0; x, z) G_y(x_0, y_0; x, y)$$

$$\times Q_p \delta(x_0 - x_S)\delta(y_0 - y_S)\delta(z_0 - z_S) \mathrm{d}z_0 \mathrm{d}y_0 \mathrm{d}x_0$$

$$= Q_p G_z(x, z; x_S, z_S) G_y(x, y; x_S, y_S)$$

where $G_z(x_0, z_0; x, z) \times G_y(x_0, y_0; x, y)$ is the Green's function that satisfies the following pairs of adjoint partial differential equations (Morse and Feshbach, 1953; Friedman, 1990) if $\alpha = \gamma$:

$$\frac{\partial}{\partial z_0}\left(bz_0^\beta \frac{\partial G_z(x_0, z_0; x, z)}{\partial z_0}\right) + az_0^\alpha \frac{\partial G_z(x_0, z_0; x, z)}{\partial x_0}$$

$$= -\delta(x_0 - x)\delta(z_0 - z) \quad \text{for } x_0 < x \tag{A1}$$

$$G_z(x_0, z_0; x, z) = 0 \quad \text{for } x_0 > x \tag{A2}$$

$$f(x_0)\frac{\partial^2 G_y(x_0, y_0; x, y)}{\partial y_0^2} + a\frac{\partial G_y(x_0, y_0; x, y)}{\partial x_0}$$

$$= -\delta(x_0 - x)\delta(y_0 - y) \quad \text{for } x_0 < x \tag{A3}$$

$$G_y(x_0, y_0; x, y) = 0 \quad \text{for } x_0 > x. \tag{A4}$$

Though physically impossible, the adjoint equations (A1)–(A4) mathematically describe, in a similar manner, the diffusion process *prior to* (i.e., as time decreases) a pollutant pulse being instantaneously applied at the location $(y_0, z_0) = (y, z)$ at a time $x_0 = x$. Alternatively, they can be viewed as steady-state turbulent diffusion with wind blowing in the *negative* x_0 direction. Thus, $G_z(x_0, z_0; x, z)$ and $G_y(x_0, y_0; x, y)$ are both zero for $x_0 > x$. Note that x_0, y_0, and z_0 have been used as independent variables instead of x, y, and z so that we can end up with $C(x, y, z)$ after integration (Greenberg, 1971).

For any location x_0, equations (A1)–(A4) above can instead be expressed in the following way, which is more amenable to solution (equations (A1) and (A2) are equivalent to (A5) and (A6), while equations (A3) and (A4) are equivalent to (A7) and (A8)):

$$\frac{\partial}{\partial z_0}\left(bz_0^\beta \frac{\partial G_z(x_0, z_0; x, z)}{\partial z_0}\right) + az_0^\alpha \frac{\partial G_z(x_0, z_0; x, z)}{\partial x_0} = 0 \tag{A5}$$

$$\lim_{x_0 \to x} az_0^\alpha G_z(x_0, z_0; x, z) = \delta(z_0 - z) \tag{A6}$$

$$f(x_0)\frac{\partial^2 G_y(x_0, y_0; x, y)}{\partial y_0^2} + a\frac{\partial G_y(x_0, y_0; x, y)}{\partial x_0} = 0 \tag{A7}$$

$$\lim_{x_0 \to x} G_y(x_0, y_0; x, y) = \delta(y_0 - y). \tag{A8}$$

Vertical sub-Green function $G_z(x_0, z_0; x, z)$

Equations (A5) and (A6) with the Neumann boundary conditions can be solved via a Laplace transform, as briefly described by Demuth (1978). The method involves the residue theorem and calculations of poles in the complex plane, which may not be familiar to readers. Here we present an alternative, better-known solution technique: separation of variables. Setting $G_z(x_0, z_0) = M(z_0)N(x_0)$, substituting into (A5) yields a pair of ordinary differential equations (A9) and (A10) with a separation constant κ^2:

$$\frac{\mathrm{d}^2 M(z_0)}{\mathrm{d}z_0^2} + \frac{\beta}{z_0}\frac{\mathrm{d}M(z_0)}{\mathrm{d}z_0} + \frac{a}{b} z_0^{\alpha-\beta}\kappa^2 M(z_0) = 0 \tag{A9}$$

$$\frac{\mathrm{d}N(x_0)}{N(x_0)} = \kappa^2 \,\mathrm{d}x_0. \tag{A10}$$

By transforming the variable for $M(z_0) = z_0^{(1-\beta)/2}\, m$ followed by $t = z_0^{(\alpha-\beta+2)/2}$, equation (A9) becomes (*A*11), whose solution consists of Bessel functions, with two constants C_1 and C_2 to be determined:

$$t^2 m'' + tm' + \frac{(a/b)\kappa^2 t^2 - [(1-\beta)/2]^2}{[(\alpha-\beta+2)/2]^2}\, m = 0 \tag{A11}$$

$$m(t) = C_1 J_\mu(\omega t) + C_2 J_{-\mu}(\omega t);$$

$$\mu = \frac{(1-\beta)/2}{(\alpha-\beta+2)/2},\quad \omega = \frac{\sqrt{(a/b)}\kappa}{(\alpha-\beta+2)/2}. \tag{A12}$$

In equation (A12), μ is written as such that it is positive for typical meteorological conditions, and is assumed to be a non-integer so that J_μ and $J_{-\mu}$ are linearly independent. The assumption can be relaxed because, as will be seen, only J_μ or $J_{-\mu}$, but not both, appears in the final solution. Substituting equation (A12) into $M(z_0)$ gives

$$M(z_0) = z_0^{(1-\beta)/2} m = z_0^{(1-\beta)/2}[C_1 J_\mu(\omega z_0^{(\alpha-\beta+2)/2}) + C_2 J_{-\mu}(\omega z_0^{(\alpha-\beta+2)/2})]. \quad (A13)$$

Equation (A10) is a first-order ordinary differential equation. Direct integration obtains its solution (C_3 is a constant):

$$N(x_0) = C_3 e^{-\kappa^2(x-x_0)} = C_3 \exp\left[-\frac{b(\alpha-\beta+2)^2\omega^2(x-x_0)}{4a}\right]. \quad (A14)$$

Combining $M(z_0)$ and $N(x_0)$ yields G_z (where C_4 and C_5 are new constants):

$$G_z(x_0, z_0) = z_0^{(1-\beta)/2}[C_4 J_\mu(\omega z_0^{(\alpha-\beta+2)/2}) + C_5 J_{-\mu}(\omega z_0^{(\alpha-\beta+2)/2})] \times \exp\left[-\frac{b(\alpha-\beta+2)^2\omega^2(x-x_0)}{4a}\right]. \quad (A15)$$

Neumann boundary type. The Neumann boundary condition at $z_0 = 0$ requires the first derivative of equation (A15) to disappear, which requires C_4 to be zero. The boundary condition at $z_0 = H$ gives the following equation:

$$\omega H^{(\alpha-\beta+2)/2} J'_{-\mu}(\omega H^{(\alpha-\beta+2)/2}) + \mu J_{-\mu}(\omega H^{(\alpha-\beta+2)/2}) = -\omega H^{(\alpha-\beta+2)/2} J_{-\mu+1}(\omega H^{(\alpha-\beta+2)/2}) = 0. \quad (A16)$$

In equation (A16), the following recurrence formula (Bowman, 1958; Tranter, 1968) has been used:

$$\tau J_{\nu+1}(\tau) = \nu J_\nu(\tau) - \tau J'_\nu(\tau). \quad (A17)$$

Therefore, the resulting eigenvalue equation becomes

$$J_{-\mu+1}(\omega H^{(\alpha-\beta+2)/2}) = 0. \quad (A18)$$

The general solution of $G_z(x_0, z_0)$ can then be expanded as the Dini series (Tranter, 1968) or the Fourier–Bessel series of the second type (Tolstov, 1976) using the eigenfunctions $J_{-\mu}$ with coefficients W_0 and W_j to be determined. The reason that initial term W_0 appears in equation (A19) is that $\omega = 0$ is a double root of equation (A16) (Bowman, 1958); i.e., it is the trivial solution to equation (A16) and is also the first root (eigenvalue) of equation (A18).

$$G_z(x_0, z_0) = W_0 + z_0^{(1-\beta)/2}\sum_{j=1}^{\infty} W_j J_{-\mu}(\omega_j z_0^{(\alpha-\beta+2)/2}) \times \exp\left[-\frac{b(\alpha-\beta+2)^2\omega_j^2(x-x_0)}{4a}\right]. \quad (A19)$$

Invoking condition (A6) gives:

$$\lim_{x_0\to x} a z_0^\alpha G_z(x_0, z_0; x, z) = a z_0^\alpha W_0 + a z_0^\alpha z_0^{(1-\beta)/2}\sum_{j=1}^{\infty} W_j J_{-\mu}(\omega_j z_0^{(\alpha-\beta+2)/2}) = \delta(z_0 - z). \quad (A20)$$

Multiplying equation (A20) by $[(\alpha-\beta+2)/2]\,dz_0$, integrating from 0 to H, and using the identity (A21) (Bowman, 1958) leaves only the W_0 term (all W_j disappear because $\sigma_i c = \omega_j H^{(\alpha-\beta+2)/2}$ are the roots of $J_{-\mu+1}$):

$$\int_0^c \tau^{\nu+1} J_\nu(\sigma_i\tau)\,d\tau = \frac{J_{\nu+1}(\sigma_i c)}{\sigma_i c} \quad (A21)$$

$$W_0 = \frac{\alpha+1}{aH^{\alpha+1}}. \quad (A22)$$

Likewise, multiplying equation (A20) by $[(\alpha-\beta+2)/2]\, z_0^{(1-\beta)/2} J_{-\mu}(\omega_i z_0^{(\alpha-\beta+2)/2})\,dz_0$, integrating from 0 to H, and using the orthogonality of the eigenfunctions given in equation (A23) (Tranter, 1968; Abramowitz and Stegun, 1970; Tolstov, 1976) yields W_j (W_0 disappears because of (A21)):

If σ is the root of $\sigma J'_\nu(\sigma c) + hJ_\nu(\sigma c) = 0$, then

$$\int_0^c \tau J_\nu(\sigma_i\tau) J_\nu(\sigma_j\tau)\,d\tau = \begin{cases} 0, & \text{if } i \neq j \\ \dfrac{1}{2\sigma_j^2}(c^2h^2 + c^2\sigma_j^2 - \nu^2)J_\nu^2(\sigma_i c), & \text{if } i = j. \end{cases} \quad (A23)$$

$$W_j = \frac{\alpha-\beta+2}{aH^{\alpha-\beta+2}} z^{(1-\beta)/2} \frac{J_{-\mu}(\omega_j z^{(\alpha-\beta+2)/2})}{J_{-\mu}^2(\omega_j H^{(\alpha-\beta+2)/2})}. \quad (A24)$$

Finally, substituting equations (A24) and (A22) into (A19), and setting $\lambda_j = \omega_j H^{(\alpha-\beta+2)/2}$ gives equation (ZR_1).

Dirichlet and mixed boundary types. The derivations for the Dirichlet boundary type is similar, except in equation (A15), C_5 disappears instead of C_4, the eigenvalue equation is equation (8) in the text instead of (A18), no W_0 term appears in equation (A19) (i.e., a Fourier–Bessel series of the first type is obtained), and the orthogonality condition is equation (A25) (Tranter, 1968; Abramowitz and Stegun, 1970; Tolstov, 1976) instead of equation (A23):

If σ is the root of $J_\nu(\sigma c) = 0$, then

$$\int_0^c \tau J_\nu(\sigma_i\tau) J_\nu(\sigma_j\tau)\,d\tau = \begin{cases} 0, & \text{if } i \neq j \\ \dfrac{c^2}{2}J_{\nu+1}^2(\sigma_j c) = \dfrac{c^2}{2}J_\nu'^2(\sigma_j c) = \dfrac{c^2}{2}J_{\nu-1}^2(\sigma_j c), & \text{if } i = j. \end{cases} \quad (A25)$$

For mixed boundary types, the solution procedure utilizes the same approach, and hence no further details are considered necessary.

Crosswind sub-Green function $G_y(x_0, y_0; x, y)$

The crosswind sub-Green function $G_y(x_0, y_0; x, y)$ can be obtained by Fourier transform. Let $\hat{G}_y(x_0, p; x, y)$ be the Fourier transformation of $G_y(x_0, y_0; x, y)$ with respect to y_0 (where p is the Fourier variable):

$$\hat{G}_y(x_0, p; x, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipy_0} G_y(x_0, y_0; x, y)\,dy_0. \quad (A26)$$

Taking the Fourier transform of equation (A7) yields a first-order ordinary differential equation (A27). Its solution by direct integration is equation (A28) with $\hat{G}_y(x_c, p; x, y)$ to be determined:

$$a\frac{d\hat{G}_y}{dx_0} - f(x_0)p^2\hat{G}_y = 0 \quad (A27)$$

$$\hat{G}_y(x_0, p; x, y) = \hat{G}_y(x_c, p; x, y) \times \exp\left[\frac{p^2}{a}\int_{x_c}^{x_0} f(\tau_0)\mathrm{d}\tau_0\right]. \quad \text{(A28)}$$

Taking the Fourier transform of equation (A8) gives (A29), which can be used to determine $\hat{G}_y(x_c, p; x, y)$:

$$\lim_{x_0 \to x} \hat{G}_y(x_0, p; x, y) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\mathrm{i}py} \quad \text{(A29)}$$

$$\hat{G}_y(x_c, p; x, y) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\mathrm{i}py} \exp\left[-\frac{p^2}{a}\int_{x_c}^{x} f(\tau_0)\mathrm{d}\tau_0\right]. \quad \text{(A30)}$$

Finally, substituting equation (A30) into (A28), taking the inverse transformation, and using the well-known formula shown as equation (A31) (Greenberg, 1971; Zauderer, 1989) gives $G_y(x_0, y_0; x, y)$ (equation (A32)):

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-\mathrm{i}p\zeta - p^2\xi)\mathrm{d}p = \frac{1}{\sqrt{4\pi\xi}}\exp\left(-\frac{\zeta^2}{4\xi}\right) \quad \text{(A31)}$$

$$\begin{aligned}
G_y(x_0, y_0; x, y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}py_0}\hat{G}_y(x_0, p)\mathrm{d}p \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}py_0}\left\{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}py} \times \exp\left[-\frac{p^2}{a}\left(\int_{x_c}^{x} f(\tau_0)\mathrm{d}\tau_0 - \int_{x_c}^{x_0} f(\tau_0)\mathrm{d}\tau_0\right)\right]\right\}\mathrm{d}p \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-\mathrm{i}p(y - y_0) - \frac{p^2}{a}\left(\int_{x_c}^{x} f(\tau_0)\mathrm{d}\tau_0 - \int_{x_c}^{x_0} f(\tau_0)\mathrm{d}\tau_0\right)\right]\mathrm{d}p \\
&= \frac{\sqrt{a}}{\sqrt{4\pi\left(\int_{x_c}^{x} f(\tau_0)\mathrm{d}\tau_0 - \int_{x_c}^{x_0} f(\tau_0)\mathrm{d}\tau_0\right)}} \times \exp\left[-\frac{a(y - y_0)^2}{4\left(\int_{x_c}^{x} f(\tau_0)\mathrm{d}\tau_0 - \int_{x_c}^{x_0} f(\tau_0)\mathrm{d}\tau_0\right)}\right].
\end{aligned} \quad \text{(A32)}$$